

# LIE MONADS AND DUALITIES

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**ABSTRACT.** We study dualities between Lie algebras and Lie coalgebras, and their respective (co)representations. To allow a study of dualities in an infinite dimensional setting, we introduce the notions of Lie monads and Lie comonads, as special cases of YB-Lie algebras and YB-Lie coalgebras in additive monoidal categories. We show that (strong) dualities between Lie algebras and Lie coalgebras are closely related to (iso)morphisms between associated Lie monads and Lie comonads. In the case of a duality between two Hopf algebras -in the sense of Takeuchi- we recover a duality between a Lie algebra and a Lie coalgebra -in the sense defined in this note- by computing the primitive elements and indecomposables.

## INTRODUCTION AND MOTIVATION

Lie coalgebras were introduced by Michaelis [12] as a formal dualization of Lie algebras. In particular, if  $(L, \Lambda)$  is a finite dimensional Lie algebra over a base field  $k$ , then the dual vector space  $C = L^*$  of  $L$  can be endowed in a natural way with the structure of a Lie coalgebra, defining the “Lie co-bracket” as the linear map  $\Upsilon = \Lambda^* : C = L^* \rightarrow C \otimes C \cong (L \otimes L)^*$ , that satisfies an anti-symmetry and “co-Jacobi” relation. Conversely, any finite dimensional Lie coalgebra gives rise to a Lie algebra on its dual space in a canonical way.

As for usual algebras and coalgebras, the passage to infinite dimensional vector spaces complicates the situation. If  $C$  is an infinite dimensional Lie coalgebra, then the dual space  $C^*$  will again be a Lie algebra. On the contrary, for an arbitrary Lie algebra  $L$ , the dual space is  $L^*$  no longer a Lie coalgebra. Rather, one should restrict to the finite dual  $L^\circ$ , which was shown -again by Michaelis- to be a Lie coalgebra. However, as we know from general considerations, the finite dual  $L^\circ$  is often too small to contain enough information to recover the complete space  $L$ . Hence, in many situations, another duality theory will be more appropriate.

The recent revival of monad theory among Hopf algebraists has shown us an alternative approach to attack these kind of dualities [2], [1]. Indeed, given a (usual) algebra  $A$  over a base field  $k$ , one can associate to it the monad  $- \otimes A$  (tensor product over  $k$ ) on the category of  $k$ -vector spaces. As the endofunctor  $- \otimes A$  has a right dual  $\text{Hom}(A, -)$ , this right dual naturally comes equipped with a comonad structure, without any finiteness condition on  $A$ . In fact, one makes the transition from algebras and coalgebras over the base field  $k$  to algebras and coalgebras in the monoidal category of endofunctors (on the category of vector spaces). The correct translation of “finite dimensionality” of vector spaces in the setting of endofunctors, is then nothing else than the existence of an adjoint functor; and a right adjoint functor for a functor of the form  $- \otimes X$  on the category of vector spaces is guaranteed by the Hom-functor  $\text{Hom}(X, -)$ .

Motivated by the above, our aim is to study a duality for Lie algebras and Lie coalgebras in such a setting. However, if we want to introduce a notion of Lie monad, we encounter a problem: the category of endofunctors is monoidal (in a canonical way), but not braided nor symmetric. Nevertheless, given a Lie algebra  $L$  or Lie coalgebra  $C$  in the category of vector spaces, one can define in a very natural way a “Lie monad” on the associated endofunctors  $- \otimes L$  and  $\text{Hom}(C, -)$ , by means of a local symmetry associated to the twist on the object  $L$  and  $C$  respectively. This leads us to the introduction of the notion of a *YB-Lie algebra* in an arbitrary additive monoidal category.

The notion of a YB-Lie algebra clearly covers the concept of a Lie algebra in a symmetric monoidal category, which in turn unifies several variations of classical Lie algebras, for example super Lie algebras. It is not our aim to go deeper into this aspect of the theory here. Instead, we refer the interested reader to the recent survey [10].

Our paper is organised as follows. After recalling some generalities on monoidal categories, we study YB-Lie algebras in Section 2. We introduce the category of Lie modules over a YB-Lie algebra and show that this is equivalent with the category of representations of the Lie algebra in case of a Lie algebra in a symmetric monoidal category. Furthermore, we study several functors and adjunctions associated to Lie modules.

In Section 3 we briefly review the dual situation of YB-Lie coalgebras and Lie comodules and provide some examples. Section 4 is devoted to the particular case of Lie monads and Lie comonads. More precisely, we show the bijective correspondence between YB-Lie algebras in an additive monoidal category and Lie monads of the form  $- \otimes L$  (see Proposition 4.4) as well as the bijective correspondence between Lie modules of a YB-Lie algebra and the (Lie version of the) Eilenberg Moore category of the associated Lie monad.

In Section 5 we start our study of dualities. We introduce the notion of a duality between a YB-Lie algebra  $L$  and YB-Lie coalgebra  $C$  in a closed monoidal category. Proposition 5.2 shows the close correspondence between dualities for the pair  $(L, C)$  and morphisms between associated Lie monads  $- \otimes L$  and  $\text{Hom}(C, -)$ , which also induces a functor between the corresponding (co)module categories. Furthermore, *strong* dualities are in correspondence with the fact that the associated Lie monad morphism is an isomorphism (Proposition 5.6), and in this situation the (co)module categories are equivalent.

It is well known that the primitive elements of a Hopf algebra form a Lie algebra. Similarly, the indecomposables of a Hopf algebra form a Lie coalgebra. Given a braided Hopf algebra, whose Yang-Baxter operator is of order two, we show in Section 6 that the primitive elements form a YB-Lie algebra in our sense, respectively the indecomposables form a YB-Lie coalgebra. Moreover, given two Hopf algebras that are in duality in the sense of Takeuchi, the associated YB-Lie algebra and YB-Lie coalgebra are in duality in our sense. Finally, we show that these dualities are in correspondence with module and comodule categories (see Theorem 6.9).

## 1. PRELIMINARIES

**Monoidal categories, braidings and symmetries.** Throughout the paper we will work in a monoidal category  $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$  with associativity constraint

$$a : \otimes \circ (\otimes \times \mathbb{1}_{\mathcal{C}}) \rightarrow \otimes \circ (\mathbb{1}_{\mathcal{C}} \times \otimes)$$

and with left- and right unit constraints resp.  $l$  and  $r$  ( $\mathbb{1}_{\mathcal{C}}$  denotes the identity functor on  $\mathcal{C}$ ). Often, if the context allows us, we will suppress the associativity and unit constraints. This will not harm the generality of our considerations, by Mac Lane's coherence theorem. In particular, all our results are applicable in situations where associativity or unit constraints are not trivial, and we will give explicit examples of these situations relevant in our setting below. Often we consider  $\mathcal{C}$  moreover to be symmetric, and denote the symmetry by  $c_{-, -}$ .

**Monoidal functors.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between monoidal categories  $(\mathcal{C}, \otimes, I)$  and  $(\mathcal{D}, \odot, J)$  is said to be a monoidal functor if it comes equipped with a natural transformation  $\phi_{X,Y} : F(X) \odot F(Y) \rightarrow F(X \otimes Y)$ ,  $X, Y \in \mathcal{C}$  and a  $\mathcal{D}$ -morphism  $\phi_0 : J \rightarrow F(I)$ , satisfying suitable compatibility conditions with relation to the associativity and unit constraints of  $\mathcal{C}$  and  $\mathcal{D}$ . Moreover,  $(F, \phi_0, \phi)$  is called a *strong* monoidal functor if  $\phi_0$  is an isomorphism and  $\phi$  is a natural isomorphism;  $(F, \phi_0, \phi)$  is called a *strict* monoidal functor  $(F, \phi_0, \phi)$ , if  $\phi_0$  is the identity morphism and  $\phi$  is the identity natural transformation.

Dually, an *op-monoidal functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a functor for which there exists a morphism  $\psi_0 : F(I) \rightarrow J$  in  $\mathcal{D}$  and morphisms  $\psi_{X,Y} : F(X \otimes Y) \rightarrow F(X) \odot F(Y)$  in  $\mathcal{D}$ , that are natural in

$X, Y \in \mathcal{C}$ , satisfying suitable compatibility conditions. Any strong monoidal functor  $(F, \phi_0, \phi)$  is automatically op-monoidal: it suffices to take  $\psi_0 = \phi_0^{-1}$  and  $\psi = \phi^{-1}$ .

If  $\mathcal{C}$  and  $\mathcal{D}$  are braided monoidal categories with braidings  $\gamma$  and  $\gamma'$  respectively, then a *braided monoidal functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a monoidal functor such that  $F\gamma_{X,Y} \circ \phi_{X,Y} = \phi_{Y,X} \circ \gamma'_{FX,FY} : FX \odot FY \rightarrow F(Y \otimes X)$ .

Of course, one has a canonical definition of a *monoidal natural transformation* between (braided) monoidal functors.

**Additivity.** Throughout,  $\mathcal{C}$  will be supposed to be an additive category, and the additive structure is compatible with the monoidal one in the usual sense, in particular  $(f + g) \otimes h = f \otimes h + g \otimes h$  for morphisms  $f, g, h$  in  $\mathcal{C}$ .

For any two object  $X, Y$  in  $\mathcal{C}$ , we denote the Hom-set from  $X$  to  $Y$  (which is supposed to be an abelian group) as  $\text{Hom}_{\mathcal{C}}(X, Y)$  or shortly by  $\text{Hom}(X, Y)$  if there can be no confusion about the category  $\mathcal{C}$ . The identity morphism on  $X$  is denoted by  $1_X$  or  $X$  for short. For any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , we denote  $\text{Id}_F$  the natural transformation defined by  $\text{Id}_{FX} = 1_{FX}$ . Although we avoid this for simplicity, most of the theory developed in this paper, can be easily extended to the setting of ( $k$ -linear) enriched categories (where  $k$  has a characteristic different from 2).

**Closedness.** Recall that a monoidal category is called *left closed* if any endofunctor of the form  $- \otimes X$  has a right adjoint. We will denote this right adjoint by  $\text{H}(X, -)$ . In this situation, for any three objects  $X, Y, Z$  in  $\mathcal{C}$ , there is an isomorphism

$$(1) \quad \pi_{Y,Z}^X : \text{Hom}(Y \otimes X, Z) \cong \text{Hom}(Y, \text{H}(X, Z)).$$

The unit and counit of the adjunction  $(- \otimes X, \text{H}(X, -))$  are denoted by

$$(2) \quad \eta_Y^X : Y \rightarrow \text{H}(X, Y \otimes X); \quad \epsilon_Y^X : \text{H}(X, Y) \otimes X \rightarrow Y.$$

One can easily observe that for a fixed object  $Y$  in  $\mathcal{C}$ , one also obtains a contravariant functor  $\text{H}(-, Y) : \mathcal{C} \rightarrow \mathcal{C}$  sending  $X$  to  $\text{H}(X, Y)$ . The functoriality comes from the fact that for any morphism  $f : X \rightarrow X'$ , one can construct

$$\text{H}(X, \epsilon_{Y'}^{X'}) \circ \text{H}(X, \text{H}(X', Y) \otimes f) \circ \eta_{\text{H}(X', Y)}^X : \text{H}(X', Y) \rightarrow \text{H}(X, Y)$$

Based on this observation, one easily obtains that  $\eta_Y^X$ ,  $\epsilon_Y^X$  and  $\pi_{Y,Z}^X$  are also natural in the argument  $X$ .

Similarly, a monoidal category is called *right closed* if any endofunctor  $X \otimes -$  has a right adjoint, that we will denote in such a situation by  $\text{H}'(X, -)$ . A monoidal category is called closed if it is both left and right closed. A braided monoidal category is closed if it is left closed or if it is right closed.

The following lemma shows that the adjunction (1) can in fact be lifted to an enriched adjunction, considering  $\mathcal{C}$  as a self-enriched category. We refer to [9, page 14] e.g. for a proof of this result.

**Lemma 1.1.** *Let  $\mathcal{C}$  be a (left) closed monoidal category, and use notation as above. Then there also exist natural isomorphisms in  $\mathcal{C}$*

$$\Pi_{Y,Z}^X : \text{H}(Y \otimes X, Z) \cong \text{H}(Y, \text{H}(X, Z))$$

Explicitly, one can compute  $\Pi$  and  $\Pi^{-1}$  in terms of  $\eta$  and  $\epsilon$ , by means of the following formulas (in the strict monoidal setting)

$$(3) \quad \Pi_{Y,Z}^X = \text{H}(Y, \text{H}(X, \epsilon_Z^{Y \otimes X})) \circ \text{H}(Y, \eta_{\text{H}(Y \otimes X, Z) \otimes Y}^X) \circ \eta_{\text{H}(Y \otimes X, Z)}^Y$$

$$(4) \quad (\Pi^{(-1)})_{Y,Z}^X = \text{H}(Y \otimes X, \epsilon_Z^X) \circ \text{H}(Y \otimes X, \epsilon_{\text{H}(X, Z)}^Y \otimes X) \circ \eta_{\text{H}(Y, \text{H}(X, Z))}^{Y \otimes X}$$

**Rigidity.** An object  $X$  in a monoidal category is called *left rigid* if there exists an object  $X^*$  together with morphisms  $\text{coev} : I \rightarrow X \otimes X^*$  and  $\text{ev} : X^* \otimes X \rightarrow I$  such that

$$(X \otimes \text{ev}) \circ a^{-1} \circ (\text{coev} \otimes X) = X, \quad (\text{ev} \otimes X^*) \circ a \circ (X^* \otimes \text{coev}) = X^*$$

It is easily verified that if  $X$  is left rigid, then the object  $X^*$  is unique up to isomorphism. In this situation, we call  $X^*$  the *left dual* of  $X$  and  $(X^*, X, \text{ev}, \text{coev})$  an *adjoint pair* in  $\mathcal{C}$ .

A *right rigid* object is defined symmetrically. Remark that if  $X$  is left rigid with left dual  $X^*$ , then  $X^*$  is right rigid with right dual  $X$ . A monoidal category is said to be *left rigid* (resp. right rigid, resp. rigid) if every object is left (resp. right, resp. both left and right) rigid. Another name for a rigid monoidal category is an *autonomous (monoidal) category*. If  $\mathcal{C}$  is braided, then it is right rigid if and only if it is left rigid. If a category is (left, right) rigid, then it is (left, right) closed and  $H(X, -) \simeq - \otimes X^*$  (resp  $H'(X, -) \simeq X^* \otimes -$ ).

**Generators.** Recall that an object  $G \in \mathcal{C}$  is called a generator if and only if the functor  $\text{Hom}_{\mathcal{C}}(G, -) : \mathcal{C} \rightarrow \underline{\text{Set}}$  is fully faithful. If the category  $\mathcal{C}$  has coproducts, this is furthermore equivalent with the fact that for any object  $X \in \mathcal{C}$  there is a canonical epimorphism  $f_X : H = \coprod_{f:G \rightarrow X} G \rightarrow X$ , where the coproduct takes over a number of copies of  $G$ . Therefore, we find a fork

$$(5) \quad \coprod_{\substack{(g,h):G \rightarrow H, \text{ st} \\ f_X \circ g = f_X \circ h}} G \xrightarrow[h_X]{g_X} \coprod_{f:G \rightarrow X} G \xrightarrow{f_X} X$$

In general this diagram is not a coequalizer, but  $G$  is called a *regular generator* if (5) is a coequalizer for every  $X \in \mathcal{C}$ , see e.g. [9, page 81].

## 2. YB-LIE ALGEBRAS AND LIE MODULES

**2.1. YB-Lie algebras in additive monoidal categories.** We will start by recalling the observation that symmetric monoidal categories are well-suited to consider cyclic permutations on copies of the same object. This result is not valid any longer on braided -rather than symmetric- monoidal categories. Consequently, the development of a theory of Lie algebras in a braided setting is a lot more involved than in the symmetric setting and leads to different possible treatments (see e.g. [14] and [11]). In this paper, we omit non-symmetric braidings, rather we allow a symmetry on an object to be a “local” gadget.

**Definition 2.1.** Let  $L$  be an object in an additive monoidal category  $\mathcal{C}$  and  $c : L \otimes L \rightarrow L \otimes L$  a morphism satisfying the following conditions:

$$(6) \quad c \circ c = \text{id}_{L \otimes L};$$

$$(7) \quad a_{L,L,L} \circ (c \otimes L) \circ a_{L,L,L}^{-1} \circ (L \otimes c) \circ a_{L,L,L} \circ (c \otimes L) \\ = (L \otimes c) \circ a_{L,L,L} \circ (c \otimes L) \circ a_{L,L,L}^{-1} \circ (L \otimes c) \circ a_{L,L,L}$$

Condition (7) is exactly the Yang-Baxter equation and (6) means that  $c$  is of order two. Hence we call a morphism  $c$  satisfying the conditions (6)-(7) a *Yang-Baxter operator of order two for  $L$* .

**Example 2.2.** If  $\mathcal{C}$  is a symmetric monoidal category, with symmetry  $c_{-, -}$ , then  $c_{L,L}$  is a Yang-Baxter operator of order two for  $L$ . Obviously,  $c_{L,L}$  satisfies conditions (6); to see that  $c_{L,L}$  also satisfies (7), one applies the hexagon condition in combination with the naturality of  $c$ .

Given an object  $L$  in  $\mathcal{C}$ , together with a Yang Baxter operator  $c$  of order two as above, we can construct the following morphisms in  $\mathcal{C}$  (compare to [5, section 5] for more details in case of Example 2.2):

$$t = t_c := a_{L,L,L} \circ (c \otimes L) \circ a_{L,L,L}^{-1} \circ (L \otimes c); \\ w = w_c := (L \otimes c) \circ a_{L,L,L} \circ (c \otimes L) \circ a_{L,L,L}^{-1}.$$

From these expressions one immediately gets that  $t \circ w = w \circ t = L$ . Let us prove the following elementary properties for these morphisms.

**Lemma 2.3.** *With notation as above, the following identities hold,*

- (i)  $t \circ t = w$ ;
- (ii)  $t \circ t \circ t = L$ .

*Proof.* (i). This is exactly (7) after reshuffling and using the invertibility of  $c$  and  $a$ .

(ii). Using now the equality  $t \circ t = w$ , we obtain immediately that  $t \circ t \circ t = t \circ w = L$ .  $\square$

*Remark 2.4.* Consider again the situation of Example 2.2. Then we can take the symmetry  $c_{L,L}$  on any object  $L$  in the symmetric monoidal category  $\mathcal{C}$ . We can construct the morphisms  $t_{c_{L,L}} = t_L$  and  $w_{c_{L,L}} = w_L$  upon which the lemma above applies. However, for general braided monoidal categories this result is no longer valid, as one can see from the following counterexample:

Let  $\mathbf{Vect}^{\mathbb{Z}_2}(k)$  denote the category whose objects are  $\mathbb{Z}_2$ -graded vector spaces over a field  $k$  ( $\text{Char}(k) \neq 2$ ), and whose morphisms consist of  $k$ -linear maps that preserve the grading. Let  $U, V, W$  be objects in  $\mathbf{Vect}^{\mathbb{Z}_2}(k)$ . Now consider the following associativity constraint  $a$  for  $\otimes_k$  (unadorned tensorproducts  $\otimes$  are to be taken over  $k$ ):

$$a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W);$$

$$(x \otimes y) \otimes z \mapsto (-1)^{|x||y||z|} x \otimes (y \otimes z),$$

where  $|x|$  denotes the degree of a homogeneous element  $x$  of an object in  $\mathbf{Vect}^{\mathbb{Z}_2}(k)$ .

Letting  $l$ , resp.  $r$  be the trivial left, resp. right unit constraints with respect to  $k$ , we obtain a (non-strict) monoidal category  $(\mathbf{Vect}^{\mathbb{Z}_2}(k), \otimes_k, k, a, l, r)$  which we shall denote by  $\mathcal{C}$ . Moreover,  $\mathcal{C}$  is a braided monoidal category if and only if  $k$  contains a primitive fourth root of unity  $i$  (see [3] for example). A braiding  $c$  can then be defined as follows; for any couple of objects  $(V, W)$  in  $\mathcal{C}$ ,

$$c_{V,W} : V \otimes W \rightarrow W \otimes V; v \otimes w \mapsto (i)^{|v||w|} w \otimes v.$$

One now checks easily that (6), and hence Lemma 2.3 does not hold for  $c = c_{V,V}$  for any object  $V$  in  $\mathcal{C}$ , with  $c$  defined as above.

**Definition 2.5.** Let  $\mathcal{C}$  be an additive, monoidal category, but not necessarily symmetric. A *YB-Lie algebra* in  $\mathcal{C}$  is a triple  $(L, \lambda, \Lambda)$ , denoted  $L$  for short if there is no confusion possible, where  $L$  is an object of  $\mathcal{C}$ ,  $\lambda$  is a Yang-Baxter operator of order two for  $L$  in  $\mathcal{C}$ , and  $\Lambda : L \otimes L \rightarrow L$  is a morphism (which we call – despite our notation – a *Lie bracket*) in  $\mathcal{C}$  that satisfies

$$(8) \quad \Lambda \circ (1_{L \otimes L} + \lambda) = 0_{L \otimes L, L},$$

$$(9) \quad \Lambda \circ (1_L \otimes \Lambda) \circ (1_{L \otimes (L \otimes L)} + t_\lambda + w_\lambda) = 0_{L \otimes (L \otimes L), L}.$$

and is such that the following diagram commutes:

$$(10) \quad \begin{array}{ccc} (L \otimes L) \otimes L & \xrightarrow{t_\lambda \circ a_{L,L,L}} & L \otimes (L \otimes L) \\ \Lambda \otimes L \downarrow & & \downarrow L \otimes \Lambda \\ L \otimes L & \xrightarrow{\lambda} & L \otimes L \end{array}$$

A morphism of YB-Lie algebras  $\phi : (L, \lambda, \Lambda) \rightarrow (L', \lambda', \Lambda')$  is a morphism  $\phi : L \rightarrow L'$  that respects the Lie-bracket, and the Yang-Baxter operator i.e.

$$(11) \quad \Lambda' \circ (\phi \otimes \phi) = \phi \circ \Lambda;$$

$$(12) \quad \lambda' \circ (\phi \otimes \phi) = (\phi \otimes \phi) \circ \lambda$$

The category of YB-Lie algebras in  $\mathcal{C}$  and morphisms of YB-Lie algebras between them is denoted by  $\text{YBLieAlg}(\mathcal{C})$ .

Suppose now that  $\mathcal{C}$  is an additive, symmetric monoidal category. A Lie algebra in  $\mathcal{C}$  is a YB-Lie algebra in  $\mathcal{C}$  of the form  $(L, c_{L,L}, \Lambda)$ , where  $c_{L,L}$  is the symmetry of the category  $\mathcal{C}$ .

The full subcategory of  $\text{YBLieAlg}(\mathcal{C})$  whose objects are Lie algebras in  $\mathcal{C}$ , is denoted by  $\text{LieAlg}(\mathcal{C})$ . Remark that a morphism between two Lie algebras automatically satisfies condition (12), by the naturality of the symmetry  $c_{-, -}$ .

We call (9) the (right)  $\lambda$ -Jacobi identity for  $L$ . As for usual Lie algebras, the definition of a YB-Lie algebra is left-right symmetric. This result follows from the following lemma that is easy to prove, an explicit proof can be found in [7].

**Lemma 2.6.** *Let  $(L, \lambda, \Lambda)$  be a YB-Lie algebra in  $\mathcal{C}$ . Then  $L$  also satisfies the left  $\lambda$ -Jacobi identity, that is the following equation holds:*

$$\Lambda \circ (\Lambda \otimes 1_L) \circ a_{L,L,L}^{-1} \circ (1_{L \otimes (L \otimes L)} + t_\lambda + w_\lambda) = 0_{L \otimes (L \otimes L), L}.$$

**Examples 2.7.** The notion of a YB-Lie algebra covers many known classes of (generalized) Lie algebras, such as: classical Lie algebras over an arbitrary commutative ring  $R$  (working in the symmetric monoidal category  $\text{Mod}(R)$ ), Lie superalgebras (working in the monoidal category of  $\mathbb{Z}_2$ -graded vector spaces, considered with the non-trivial symmetry) and certain classes of Hom-Lie algebras (applying the Hom-construction on an additive symmetric monoidal category, see [5] for more details about this non-strict example). For more details about the examples above, we refer to [7]. It also covers the theory of Lie monads (working in a (non-symmetric) monoidal category of additive endofunctors on an additive category) -treated in more detail in Section 4.1- and in Section 6.1 the YB-Lie algebra of primitive elements of a braided bialgebra is constructed.

Finally, one observes that if  $(L, \lambda, \Lambda)$  is a Lie algebra, then  $(L, \lambda, \Lambda \circ \lambda)$  is again a YB-Lie algebra, which we call the opposite Lie algebra of  $L$ .

**2.2. Lie modules.** Let  $(\mathcal{C}, \otimes, I, a, l, r)$  be an additive, monoidal category and let  $(L, \lambda, \Lambda)$  be a YB-Lie algebra in  $\mathcal{C}$ .

**Definition 2.8.** A *right Lie module over  $L$*  is an object  $X$  in  $\mathcal{C}$ , together with a morphism  $\varrho : X \otimes L \rightarrow X$  satisfying

$$(13) \quad \varrho \circ \left( (\varrho \otimes L) \circ a_{X,L,L}^{-1} - (\varrho \otimes L) \circ a_{X,L,L}^{-1} \circ (X \otimes \lambda) - X \otimes \Lambda \right) = 0_{X \otimes (L \otimes L), X}.$$

Left Lie modules can be introduced symmetrically.

**Example 2.9.** Let  $(L, \lambda, \Lambda)$  be a YB-Lie algebra in  $\mathcal{C}$ . Then  $L$  is a Lie module over itself (with  $\varrho = \Lambda$ ). One easily gets (13) from the Jacobi identity and antisymmetry.

**Definition 2.10.** Let  $(X, \varrho_X)$  and  $(Y, \varrho_Y)$  be two right Lie modules in  $\mathcal{C}$ . A morphism of Lie modules is a morphism  $f : X \rightarrow Y$  s.t. the following diagram commutes

$$\begin{array}{ccc} X \otimes L & \xrightarrow{\varrho_X} & X \\ f \otimes L \downarrow & & \downarrow f \\ Y \otimes L & \xrightarrow{\varrho_Y} & Y \end{array}$$

The set of all morphisms of Lie modules from  $X$  to  $Y$  is denoted by  $\text{LHom}(X, Y)$ . Then, with these definitions, Lie modules in a monoidal category  $\mathcal{C}$  together with their morphisms form a category, which we will denote by  $\text{LieMod}(L)$  (whether we consider left or right modules is supposed to be clear from the context).

*Remark 2.11.* If  $L$  is a Lie algebra in a symmetric monoidal category, then the category of left Lie modules over  $L$  and right Lie modules over  $L$  are isomorphic.

**Definition 2.12.** Let  $\mathcal{C}$  be an additive, symmetric closed monoidal category. A *representation of a Lie algebra*  $(L, \Lambda_L)$  is a pair  $(X, \phi_X)$ , where  $X$  is an object of  $\mathcal{C}$  and  $\phi_X : (L, \Lambda_L) \rightarrow (\mathbf{H}(X, X), \Lambda_{\mathbf{H}(X, X)})$  is a morphism of Lie algebras, where  $\Lambda_{\mathbf{H}(X, X)}$  is the commutator Lie bracket, defined as follows:  $\Lambda_{\mathbf{H}(X, X)} = m_{\mathbf{H}(X, X)} - m_{\mathbf{H}(X, X)} \circ c_{\mathbf{H}(X, X), \mathbf{H}(X, X)}$ , with

$$m_{\mathbf{H}(X, X)} = \pi_{\mathbf{H}(X, X) \otimes \mathbf{H}(X, X), X}^X (\epsilon_X^X \circ (\mathbf{H}(X, X) \otimes \epsilon_X^X)).$$

Morphisms are defined as follows: Let  $(X, \phi_X)$  and  $(Y, \phi_Y)$  be two representations of  $(L, \Lambda_L)$  and let  $f : X \rightarrow Y$  a morphism in  $\mathcal{C}$ . Then  $f$  is a morphism of representations if the following diagram commutes

$$\begin{array}{ccccc} L \otimes X & \xrightarrow{\phi_X \otimes X} & \mathbf{H}(X, X) \otimes X & \xrightarrow{\epsilon_X^X} & X \\ L \otimes f \downarrow & & & & \downarrow f \\ L \otimes Y & \xrightarrow{\phi_Y \otimes Y} & \mathbf{H}(Y, Y) \otimes Y & \xrightarrow{\epsilon_Y^Y} & Y \end{array}$$

The category of representations of  $L$  is denoted by  $\text{Rep}(L)$ .

**Proposition 2.13.** *Let  $L$  be a Lie algebra in a symmetric closed monoidal category. There is an equivalence of categories between the category of (left) Lie modules  $\text{LieMod}(L)$  and the category of representations  $\text{Rep}(L)$ .*

*Proof.* We define a functor  $F : \text{LieMod}(L) \rightarrow \text{Rep}(L)$  as follows:

$$F(X, \varrho_X) = (X, \phi_X = \pi_{L, X}^X(\varrho_X)),$$

for any (left) Lie module  $(X, \varrho_X)$ , and  $F$  acts as the identity functor on morphisms. By naturality of  $\epsilon_X^X$ , we have  $\epsilon_X^X(\pi_{L, X}^X(\varrho_X) \otimes X) = \varrho_X$ . Applying this together with the naturality of  $\pi$ , one can check that  $F$  is well-defined.

Conversely, consider the functor  $G : \text{Rep}(L) \rightarrow \text{LieMod}(L)$  defined for any object  $(X, \phi_X)$  of  $\text{Rep}(L)$  as

$$G(X, \phi_X) = (X, \varrho_X = (\pi_{L, X}^X)^{-1}(\phi_X)),$$

and  $G$  is the identity on morphisms. To see that  $G$  is well-defined, it suffices to make use of the naturality of  $c$ ,  $\epsilon_X^X$  and  $\pi_{L, X}^X$ .

Finally, it is clear that  $(G, F)$  is pair of adjoint functors with trivial unit and counit (i.e. identical natural transformations), hence they establish the desired equivalence of categories.  $\square$

### 2.3. Adjoint functors for Lie modules. Example 2.9 leads to the following

**Proposition 2.14.** *Let  $(L, \lambda, \Lambda)$  be a YB-Lie algebra in an additive monoidal category  $\mathcal{C}$  and  $(M, \varrho_M)$  a Lie module. Then there is a functor*

$$- \otimes M : \mathcal{C} \rightarrow \text{LieMod}(L).$$

*In particular, there is a functor*

$$- \otimes L : \mathcal{C} \rightarrow \text{LieMod}(L).$$

*Proof.* Let  $X$  be any object of  $\mathcal{C}$ , then the Lie module-structure on  $X \otimes M$  is defined by  $\varrho_{X \otimes M} = X \otimes \varrho_M$ . It is easily checked that this indeed defines a functor.

Applying this for the Lie module  $(L, \Lambda)$ , we obtain the functor  $- \otimes L$ .  $\square$

A natural question that arises is whether these functors have a right adjoint. To obtain this result, we need to shift our setting towards *closed* monoidal categories. In the remaining of this section, we will suppose that  $\mathcal{C}$  is an additive, left closed monoidal category.

The proof of the following theorem is based on the observation that the set of morphisms between  $L$ -Lie modules can be expressed as the following equalizer: Let  $(M, \varrho_M)$  and  $(N, \varrho_N)$  be two  $L$ -Lie modules, then we have the following equalizer in Ab

$$\mathrm{LHom}(M, N) \longrightarrow \mathrm{Hom}(M, N) \begin{array}{c} \xrightarrow{(-) \circ \varrho_M} \\ \xrightarrow{\varrho_N \circ ((-) \otimes L)} \end{array} \mathrm{Hom}(M \otimes L, N)$$

To obtain a right adjoint for the functor of Proposition 2.14, we need to lift this equalizer to the category  $\mathcal{C}$ .

**Theorem 2.15.** *Suppose that  $\mathcal{C}$  possesses equalizers. Then the functor  $- \otimes M : \mathcal{C} \rightarrow \mathrm{LieMod}(L)$  has a right adjoint  $\mathrm{LH}(M, -)$ , given by the following equalizer in  $\mathcal{C}$*

$$(14) \quad \mathrm{LH}(M, N) \longrightarrow \mathrm{H}(M, N) \begin{array}{c} \xrightarrow{\pi_{\mathrm{H}(M, N), N}^{M \otimes L}(\epsilon_N^M \circ (1_{\mathrm{H}(M, N)} \otimes \varrho_M))} \\ \xrightarrow{\pi_{\mathrm{H}(M, N), N}^{M \otimes L}(\varrho_N \circ (\epsilon_N^M \otimes L))} \end{array} \mathrm{H}(M \otimes L, N)$$

for any Lie module  $(N, \varrho_N)$ .

*Proof.* We have to proof that there is a natural isomorphism  $\mathrm{LHom}(X \otimes M, N) \cong \mathrm{Hom}(X, \mathrm{LH}(M, N))$  for any object  $X \in \mathcal{C}$  and any  $L$ -Lie module  $(N, \varrho_N)$ .

Consider the following equalizer in Ab:

$$\mathrm{LHom}(X \otimes M, N) \longrightarrow \mathrm{Hom}(X \otimes M, N) \begin{array}{c} \xrightarrow{(-) \circ (X \otimes \varrho_M)} \\ \xrightarrow{\varrho_N \circ ((-) \otimes L)} \end{array} \mathrm{Hom}(X \otimes M \otimes L, N)$$

Recall (cf. e.g [4, Proposition 2.9.4]) that a representable functor preserves all limits. Hence if we apply the representable functor  $\mathrm{Hom}(X, -)$  on the equalizer (14) defining  $\mathrm{LH}(M, N)$ , we obtain the following equalizer in Ab:

$$\mathrm{Hom}(X, \mathrm{LH}(M, N)) \longrightarrow \mathrm{Hom}(X, \mathrm{H}(M, N)) \begin{array}{c} \xrightarrow{\left(\pi_{\mathrm{H}(M, N), N}^{M \otimes L}(\epsilon_N^M \circ (1_{\mathrm{H}(M, N)} \otimes \varrho_M))\right)^*} \\ \xrightarrow{\left(\pi_{\mathrm{H}(M, N), N}^{M \otimes L}(\varrho_N \circ (\epsilon_N^M \otimes L))\right)^*} \end{array} \mathrm{Hom}(X, \mathrm{H}(M \otimes L, N)),$$

where  $(-)^*$  denotes  $\mathrm{Hom}(X, -)$ . We know that  $\pi_{X, N}^M$  and  $\pi_{X, N}^{M \otimes L}$  respectively provide isomorphisms between the last two objects in the above two equalizers. Our aim is now to show that these isomorphisms induce an isomorphism between the respective equalizers. Take  $f \in \mathrm{Hom}(X \otimes M, N)$ , then we find

$$\begin{aligned} & \left(\pi_{\mathrm{H}(M, N), N}^{M \otimes L}(\epsilon_N^M \circ (\mathrm{H}(M, N) \otimes \varrho_M))\right)^* \circ \pi_{X, N}^M(f) \\ &= \pi_{X, N}^{M \otimes L}(\epsilon_N^M \circ (\mathrm{H}(M, N) \otimes \varrho_M) \circ (\pi_{X, N}^M(f) \otimes M \otimes L)) \\ &= \pi_{X, N}^{M \otimes L}(\epsilon_N^M \circ (\pi_{X, N}^M(f) \otimes M) \circ (X \otimes \varrho_M)) \\ &= \pi_{X, N}^{M \otimes L}(f \circ (X \otimes \varrho_M)) \end{aligned}$$

where we used the naturality of  $\pi_{-, N}^{M \otimes L}$  in the first equality and the naturality of the tensor product in the second equality and the naturality of  $\epsilon_-^M$  in combination with the fact that  $\epsilon_{X \otimes M}^M \circ (\eta_X^M \otimes M) = 1_{X \otimes M}$  and  $\pi_{X, N}^M = \mathrm{H}(M, f) \circ \eta_Y^M$  in the last equality.

A similar computation shows that

$$\pi_{X, N}^{M \otimes L}(\varrho_N \circ (f \otimes L)) = \pi_{\mathrm{H}(M, N), N}^{M \otimes L}(\varrho_N \circ (\epsilon_N^M \otimes L)) \circ \pi_{X, N}^M(f).$$

Now, by the uniqueness of the equalizer, we obtain a natural isomorphism  $\mathrm{LHom}(X \otimes M, N) \cong \mathrm{Hom}(X, \mathrm{LH}(M, N))$ , which shows the adjunction between  $- \otimes M$  and  $\mathrm{LH}(M, -)$ .  $\square$



**Construction 2.16. The commutator Lie algebra** Let  $(B, \mu_B)$  be a (non-unital) associative algebra in  $\mathcal{C}$ . We say that  $B$  is a YB-algebra if it comes equipped with a self-invertible Yang-Baxter operator  $\lambda_B : B \otimes B \rightarrow B \otimes B$  that satisfies the following condition

$$(15) \quad \begin{array}{ccccc} B \otimes B \otimes B & \xrightarrow{w_{\lambda_B}} & B \otimes B \otimes B & \xrightarrow{t_{\lambda_B}} & B \otimes B \otimes B \\ \mu_B \otimes B \downarrow & & B \otimes \mu_B \downarrow & & \mu_B \otimes B \downarrow \\ B \otimes B & \xrightarrow{\lambda_B} & B \otimes B & \xrightarrow{\lambda_B} & B \otimes B \end{array}$$

The category of YB-algebras in  $\mathcal{C}$  is denoted by  $\mathbf{YBAlg}(\mathcal{C})$ . Then there is a functor

$$\mathcal{L} : \mathbf{YBAlg}(\mathcal{C}) \rightarrow \mathbf{YBLieAlg}(\mathcal{C})$$

that sends any YB-algebra  $(B, \mu_B, \lambda_B)$  to the YB-Lie algebra  $(B, \Lambda_B, \lambda_B)$  with commutator Lie-bracket  $\Lambda_B = \mu_B \circ (B \otimes B - \lambda_B)$ .

Let us fix a YB-Algebra  $B$  and denote the category of (right)  $B$ -modules  $(M, \rho_M)$ ,  $(\rho_M$  being the right action of  $B$  on  $M$ ) by  $\mathbf{Mod}(B)$ . Then we can define a functor

$$\underline{\mathbf{Ind}}(-) : \mathbf{Mod}(B) \longrightarrow \mathbf{LieMod}(\mathcal{L}(B))$$

by putting  $\underline{\mathbf{Ind}}(M, \rho_M) = (M, \varrho_M = \rho_M)$ . The fact that  $\underline{\mathbf{Ind}}(-)$  is well-defined follows from the (mixed) associativity of the (right) action of  $B$  onto any (right)  $B$ -module. Remark that because of this, a YB-algebra  $B$  always possesses two  $\mathcal{L}(B)$ -Lie module structures: one by its commutator Lie-bracket, and one by its initial (associative) multiplication.

We will search for an adjoint for the functor  $\underline{\mathbf{Ind}}$ . However, we will work in a more general setting. Let  $(L, \lambda, \Lambda)$  be any YB-Lie algebra and  $B$  an associative algebra. Let  $(T, \varrho_T)$  be a  $L$ -Lie module that is at the same time a left  $B$ -module with action  $m : B \otimes T \rightarrow T$  such that  $m$  is a morphism of  $L$ -Lie modules, where the  $L$ -Lie module structure on  $B \otimes T$  is given by  $B \otimes \varrho_T$ , i.e. it is the structure induced by the functor  $- \otimes T$  of Proposition 2.14 evaluated in  $B$ . This means that the following diagram commutes:

$$(16) \quad \begin{array}{ccc} B \otimes T \otimes L & \xrightarrow{B \otimes \varrho_T} & B \otimes T \\ m \otimes L \downarrow & & m \downarrow \\ T \otimes L & \xrightarrow{\varrho_T} & T \end{array}$$

Then there is a well-defined functor

$$- \otimes_B T : \mathbf{Mod}(B) \rightarrow \mathbf{LieMod}(L).$$

In case we take  $L = \mathcal{L}(B)$ , and  $(T, \varrho_T) = (B, \mu_B)$  with the regular left  $B$ -action, then we find that this functor is exactly  $\underline{\mathbf{Ind}}$ . Before stating the next theorem, we need two little lemmas:

**Lemma 2.17.** *Suppose  $\mathcal{C}$  to be complete and let  $B$  be an associative algebra in  $\mathcal{C}$ . Then the forgetful functor  $\mathcal{U} : \mathbf{Mod}(B) \rightarrow \mathcal{C}$  reflects limits.*

*Proof.* As  $\mathcal{C}$  is complete, we know that  $\mathbf{Mod}(B)$  is complete as well (see e.g. [15, Fact 2]). Now, we observe that  $\mathcal{U}$  preserves limits, since it has a left adjoint  $- \otimes B$ . Finally, we also have that  $\mathcal{U}$  reflects isomorphisms. The lemma now follows from e.g. [4, Proposition 2.9.7].  $\square$

We are now ready to proof the following

**Theorem 2.18.** *Suppose that  $\mathcal{C}$  is an additive, left closed monoidal category with equalizers. Let  $(L, \lambda, \Lambda)$  be a YB-Lie algebra in  $\mathcal{C}$ ,  $B$  an associative algebra in  $\mathcal{C}$ ,  $(T, \varrho_T)$  a  $L$ -Lie module that is a left  $B$ -module with action  $m$  such that (16) commutes. Then then the functor  $\mathbf{LH}(T, -)$  from Theorem 2.15 can be corestricted to obtain a right adjoint to the above defined functor*

$$- \otimes_B T : \mathbf{Mod}(B) \rightarrow \mathbf{LieMod}(L).$$

*Proof.* Let  $(M, \varrho_M)$  be an object in  $\text{LieMod}(L)$ . Define  $\text{LH}(T, (M, \varrho_M)) = (\text{LH}(T, M), \rho_M)$ , with

$$\rho_M = \text{LH}(T, \zeta_M \circ (\text{LH}(T, M) \otimes m)) \circ \theta_{\text{LH}(T, M) \otimes B},$$

where  $\theta_X : X \rightarrow \text{LH}(T, X \otimes T)$  is the unit and  $\zeta_M : \text{LH}(T, M) \otimes T \rightarrow M$  the counit of the adjunction between  $- \otimes T$  and  $\text{LH}(T, -)$  from Theorem 2.15. Then it follows smoothly, from naturality and the fact that  $m$  is a left  $B$ -action on  $T$ , that  $\rho_M$  defines an associative and unital right  $B$ -action.

To prove the adjunction, we need to prove that we have an isomorphism of abelian groups

$$\text{LHom}(M \otimes_B T, N) \cong \text{Hom}_B(M, \text{LH}(T, N)),$$

whenever  $(N, \varrho_N) \in \text{LieMod}(L)$  and  $(M, \rho_M) \in \text{Mod}(B)$ . To this end, we use a similar argument as in Theorem 2.15. First, remark that Lie module homomorphisms from  $M \otimes_B T$  to  $N$  can be characterized as the following equalizer in Ab:

$$\text{LHom}(M \otimes_B T, N) \longrightarrow \text{Hom}(M \otimes_B T, N) \xrightleftharpoons[\varrho_N \circ ((-) \otimes L)]{(-) \circ (M \otimes_B \varrho_T)} \text{Hom}(M \otimes_B T \otimes L, N).$$

Next, we consider the following equalizer in  $\mathcal{C}$ :

$$(17) \quad \text{LH}(T, N) \longrightarrow \text{H}(T, N) \xrightleftharpoons[\pi_{\text{H}(T, N), N}^{T \otimes L}(\varrho_N \circ (\epsilon_N^T \otimes L))]{\pi_{\text{H}(T, N), N}^{T \otimes L}(\epsilon_N^T \circ (id_{\text{H}(T, N)} \otimes_B \varrho_T))} \text{H}(T \otimes L, N),$$

where  $\pi$  and  $\epsilon$  denote as before the (natural) isomorphisms associated to the adjunction between  $- \otimes T$  and  $\text{H}(T, -)$ . We know from the first part of the proof that  $\text{LH}(T, N)$  is moreover a right  $B$ -module. In a similar way, classical arguments of enriched category theory tell us that  $\text{H}(T, N)$  and  $\text{H}(T \otimes L, N)$  are right  $B$ -modules (we even have an adjunction  $(- \otimes_B T, \text{H}(T, -))$  between  $\text{Mod}(B)$  and  $\mathcal{C}$ ). Hence, (17) is an equalizer in  $\text{Mod}(B)$  by Lemma 2.17. We can thus apply the representable functor  $\text{Hom}_B(M, -)$  to this equalizer, to obtain the following equalizer in Ab.

$$\text{Hom}_B(M, \text{LH}(T, N)) \longrightarrow \text{Hom}_B(M, \text{H}(T, N)) \xrightleftharpoons[\left(\pi_{\text{H}(T, N), N}^{T \otimes L}(\varrho_N \circ (\epsilon_N^T \otimes L))\right)^*]{\left(\pi_{\text{H}(T, N), N}^{T \otimes L}(\epsilon_N^T \circ (id_{\text{H}(T, N)} \otimes_B \varrho_T))\right)^*} \text{Hom}_B(M, \text{H}(T \otimes L, N))$$

To conclude the proof, it now suffices to observe that by the adjunction  $(- \otimes_B T, \text{H}(T, -))$ , we have isomorphisms  $\text{Hom}(M \otimes_B T, N) \cong \text{Hom}_B(M, \text{H}(T, N))$  and  $\text{Hom}(M \otimes_B T \otimes L, N) \cong \text{Hom}_B(M, \text{H}(T \otimes L, N))$ , which implies that the above constructed equalizers in Ab are isomorphic.  $\square$

As a particular instance of Theorem 2.18, we find that the functor  $\underline{\text{Ind}} : \text{Mod}(B) \rightarrow \text{LieMod}(B)$ , being naturally isomorphic to  $- \otimes_B B$ , has a right adjoint  $\text{LH}(B, -)$ . Moreover, we obtain the following diagram of adjoint functors for any right  $B$ -module  $(T, \rho_T)$ .

$$\begin{array}{ccc} & \mathcal{C} & \\ \begin{array}{c} \swarrow - \otimes T \\ \searrow \text{H}_B(T, -) \end{array} & & \begin{array}{c} \nwarrow - \otimes \hat{T} \\ \swarrow \text{LH}(\hat{T}, -) \end{array} \\ \text{Mod}(B) & \xrightleftharpoons[\underline{\text{Ind}} \simeq - \otimes_B B]{\text{LH}(B, -)} & \text{LieMod}(\mathcal{L}(B)) \end{array}$$

Here we denote  $\hat{T} = \underline{\text{Ind}}(T)$ , the induced  $L$ -Lie module of  $T$  and the functor  $\text{H}_B(T, -) : \text{Mod}(B) \rightarrow \mathcal{C}$  is the internal representable functor defined by the equalizer for all  $(X, \rho_X) \in$

$\text{Mod}(B)$

$$\mathbf{H}_B(T, X) \longrightarrow \mathbf{H}(T, X) \xrightarrow[\pi_{\mathbf{H}(T, X), X}^{T \otimes B}(\rho_X \circ (\epsilon_X^T \otimes B))]{\pi_{\mathbf{H}(T, X), X}^{T \otimes B}(\epsilon_X^T \circ (1_{\mathbf{H}(T, X)} \otimes \rho_T))} \mathbf{H}(T \otimes B, X),$$

which is known to be a right adjoint for  $- \otimes T : \mathcal{C} \rightarrow \text{Mod}(B)$ . Clearly  $\underline{\text{Ind}}(X \otimes T) \cong X \otimes \hat{T}$ , so the outer triangle in the above diagram naturally commutes. To see that the inner diagram of functors also commutes, take any object  $X$  in  $\text{LieMod}(\mathcal{L}(B))$  and  $Y$  in  $\mathcal{C}$ . By applying the adjunctions above, we then find

$$\begin{aligned} \text{Hom}(Y, \mathbf{H}_B(T, \text{LH}(B, X))) &\cong \text{Hom}_B(Y \otimes T, \text{LH}(B, X)) \cong \text{LHom}(\underline{\text{Ind}}(Y \otimes B), X) \\ &\cong \text{LHom}(Y \otimes \hat{T}, X) \cong \text{Hom}(Y, \text{LH}(\hat{T}, X)) \end{aligned}$$

So by the Yoneda lemma, we find a natural isomorphism  $\mathbf{H}_B(T, \text{LH}(B, X)) \cong \text{LH}(\hat{T}, X)$ .

### 3. YB-LIE COALGEBRAS AND LIE COMODULES

Let  $(\mathcal{C}, \otimes, I)$  be an additive, monoidal category. *YB-Lie coalgebras in  $\mathcal{C}$*  are defined dually to YB-Lie algebras, i.e. we define the category  $\text{YBLieCoAlg}(\mathcal{C})$  of YB-Lie coalgebras in  $\mathcal{C}$  as  $\text{YBLieCoAlg}(\mathcal{C}) = \text{YBLieAlg}(\mathcal{C}^{op})^{op}$ . If  $\mathcal{C}$  is moreover symmetric, we define Lie coalgebras in  $\mathcal{C}$  by  $\text{LieCoAlg}(\mathcal{C}) = \text{LieAlg}(\mathcal{C}^{op})^{op}$ . Here  $\mathcal{C}^{op} = (\mathcal{C}^{op}, \otimes^{op}, I)$  denotes the opposite category of  $\mathcal{C}$  and  $\otimes^{op} : \mathcal{C}^{op} \times \mathcal{C}^{op} \rightarrow \mathcal{C}^{op}$  the opposite tensor product functor induced in the obvious way by  $\otimes$ . Explicitly, this leads to the following definition, which is due to Michaelis in the symmetric case (cf. [12]).

**Definition 3.1.** A YB-Lie coalgebra in  $\mathcal{C}$  is a triple  $(C, \gamma, \Upsilon)$ , denoted  $C$  for short if no confusion can be made, consisting of an object  $C$  in  $\mathcal{C}$  together with a self-invertible YB-operator  $\gamma : C \otimes C \rightarrow C \otimes C$  and a comultiplication map  $\Upsilon : C \rightarrow C \otimes C$  such that

- (1)  $\Upsilon + \gamma \circ \Upsilon = 0_{C, C \otimes C}$ ;
- (2)  $(1_{C \otimes (C \otimes C)} + w_\gamma + t_\gamma) \circ (1_C \otimes \Upsilon) \circ \Upsilon = 0_{C, C \otimes (C \otimes C)}$ .

and is such that the following diagram commutes:

$$\begin{array}{ccc} C \otimes C & \xrightarrow{\gamma} & C \otimes C \\ C \otimes \Upsilon \downarrow & & \downarrow \Upsilon \otimes C \\ C \otimes (C \otimes C) & \xrightarrow{a_{C, C, C}^{-1} \circ w_\gamma} & (C \otimes C) \otimes C \end{array}$$

In case that  $\mathcal{C}$  is moreover symmetric, then we call a YB-Lie coalgebra of the form  $(C, c_{C, C}, \Upsilon)$ , where  $c_{C, C}$  is the symmetry of  $\mathcal{C}$ , just a Lie-coalgebra.

For the sake of completeness, let us write down the following

**Definition 3.2.** A *Lie comodule over  $C$*  is an object  $X$  in  $\mathcal{C}$ , endowed with a morphism  $\delta^X : X \rightarrow X \otimes C$  satisfying

$$\left( a_{X, C, C} \circ (\delta^X \otimes C) - (X \otimes \gamma) \circ a_{X, C, C} \circ (\delta^X \otimes C) - X \otimes \Upsilon \right) \circ \delta^X = 0_{X, X \otimes (C \otimes C)}.$$

Morphisms of Lie comodules are defined in the obvious way. The category of Lie comodules over  $C$  with their morphisms will be denoted by  $\text{LieCoMod}(C)$ .

All statements and theorems of the previous section have obvious duals for Lie coalgebras and Lie comodules. There is no point in repeating these explicitly. Let us just finish this section by mentioning some examples (see also [12] for Example 3.3 (1),(2) and (4)) of Lie coalgebras that will be useful later on.

**Examples 3.3.** (1) Let  $(C, \Delta_C)$  be a coassociative coalgebra in an additive, monoidal category  $\mathcal{C}$  and suppose there is a self-invertible Yang-Baxter operator  $\gamma : C \otimes C \rightarrow C \otimes C$  on  $C$ , such that an analogous version of the diagram (15) commutes, then we can consider a YB-Lie coalgebra structure on  $\mathcal{L}^c(C) = C$ , defined by  $\Upsilon_{\mathcal{L}^c(C)} = (C \otimes C - \gamma) \circ \Delta_C$ .

- (2) Let  $H$  be a Hopf algebra in  $\text{Vect}(k)$ . Let  $I = \text{Ker}(\epsilon)$ , with  $\epsilon$  the counit of  $H$ , and let us denote  $Q(H) = I/I^2$ , the so-called *indecomposables* of  $H$ . Then  $Q(H)$  is a Lie coalgebra, where the cobracket comes from  $\Delta_{\mathcal{L}^c(H)}$ . To see that this is true, let us first check that  $\Delta_{\mathcal{L}^c(H)} : I \rightarrow I \otimes I$  is well-defined. Indeed, since

$$\text{Im}(\Delta_{\mathcal{L}^c(I)}) \subset \text{Ker}(\epsilon \otimes 1_H) \cap \text{Ker}(1_H \otimes \epsilon) = (I \otimes H) \cap (H \otimes I) = I \otimes I,$$

where we denoted  $\Delta_{\mathcal{L}^c(I)}$  the restriction of  $\Delta_{\mathcal{L}^c(H)}$  to  $I$ . Moreover,  $\Delta_{\mathcal{L}^c(I)}$  maps  $I^2$  into  $I^2 \otimes I + I \otimes I^2$ , so the map

$$\overline{\Delta_{\mathcal{L}^c(I)}} : I/I^2 \rightarrow I \otimes I / (I^2 \otimes I + I \otimes I^2) \cong I/I^2 \otimes I/I^2$$

is well-defined and turns  $Q(H)$  into a Lie coalgebra. Let us point out that dually to the Lie algebra case,  $Q(H)$  can be described as the following coequalizer

$$H \otimes H \xrightleftharpoons[\epsilon \otimes H + H \otimes \epsilon]{\mu} H \longrightarrow Q(H)$$

Let us remark that the construction of indecomposables in terms of a coequalizer as above, allows to perform this construction in any category with sufficiently well-behaving coequalizers. We will come back to this in Section 6.1.

- (3) The next example is closely related to the previous one. Let  $H$  be again a Hopf algebra. Consider the space  $X \subset H$  consisting of all  $x$  such that

$$(18) \quad (f * g)(x) = f(x) + g(x),$$

where  $f, g \in H^*$  and  $*$  is the convolution product. Then one can compute that the comultiplication restricted to  $X$  is cocommutative (that is  $\tau \circ \Delta = \Delta$  on all elements of  $X$ , where  $\tau : H \otimes H \rightarrow H \otimes H$  is the switch map). Indeed: consider a base  $\{e_i\}$  for  $X$ , then  $x = \sum a_i e_i$ ,  $\Delta(x) = \sum a_{ij} e_i \otimes e_j$ . Now apply condition (18) for the dual base elements  $f = e_i^*$  and  $g = e_j^*$ , then one finds that  $a_{ij} = a_i + a_j$ . The cocommutativity now follows. We consider the quotient space  $C = H/X$ . Since  $X$  is cocommutative, the map  $\Upsilon = \Delta - \tau \circ \Delta$  is well-defined on  $C$ . One can now check that  $(C, \Upsilon)$  is a Lie-coalgebra. Let us call  $C$  the *Lie coalgebra of coprimitives*. Moreover, there is a Lie coalgebra morphism

$$C \rightarrow Q(H); \quad \bar{h} \mapsto \overline{h - \eta(\epsilon(h))}$$

- (4) Let  $L$  be a finite dimensional Lie  $k$ -algebra. Then its dual space  $C = L^*$  can be endowed with the structure of a Lie coalgebra, by putting  $\Upsilon : L^* \rightarrow L^* \otimes L^* \cong (L \otimes L)^*$ ; the dual map of the Lie bracket. Similarly, if  $L$  is a YB-Lie algebra, then  $C$  is a YB-Lie coalgebra with  $\gamma = \lambda^*$ . Conversely, if  $C$  is any Lie coalgebra (or YB-Lie coalgebra), even infinite dimensional, then its dual space  $C^*$  becomes a Lie algebra. We will treat this in more detail in Section 5.
- (5) Let  $\mathcal{A}$  be any additive category, and  $\text{End}(\mathcal{A})$  an additive, monoidal category of additive endofunctors on  $\mathcal{A}$  and natural transformations between them. We will call a YB-Lie coalgebra in  $\text{End}(\mathcal{A})$  a *Lie comonad* on  $\mathcal{A}$ , see Remark 4.7.

#### 4. LIE MONADS AND COMONADS

**4.1. Lie monads.** We already introduced Lie monads in Example 2.7 as YB-Lie algebras in a category of additive endofunctors on an additive category, let us restate the definition in explicit form.

**Definition 4.1.** A *Lie monad* on an additive category  $\mathcal{C}$  is a triple  $(\mathbf{L}, \lambda, \Lambda)$ , where  $\mathbf{L} : \mathcal{C} \rightarrow \mathcal{C}$  is an additive functor,  $\lambda : \mathbf{L} \circ \mathbf{L} \rightarrow \mathbf{L} \circ \mathbf{L}$  is a natural transformation satisfying the self-invertible Yang-Baxter equations

$$(19) \quad \lambda \circ \lambda = \text{id}_{\mathbf{L} \circ \mathbf{L}}$$

$$(20) \quad (\text{id}_{\mathbf{L}} * \lambda) \circ (\lambda * \text{id}_{\mathbf{L}}) \circ (\text{id}_{\mathbf{L}} * \lambda) = (\lambda * \text{id}_{\mathbf{L}}) \circ (\text{id}_{\mathbf{L}} * \lambda) \circ (\lambda * \text{id}_{\mathbf{L}})$$

and  $\Lambda : \mathbf{L} \circ \mathbf{L} \rightarrow \mathbf{L}$  is a natural transformation satisfying the following conditions:

$$(21) \quad \Lambda \circ (\text{id}_{\mathbf{L} \circ \mathbf{L}} + \lambda) = 0_{\mathbf{L} \circ \mathbf{L}, \mathbf{L}}$$

$$(22) \quad \Lambda \circ (\Lambda * \text{id}_{\mathbf{L}}) \circ (\text{id}_{\mathbf{L} \circ \mathbf{L}} + t_\lambda + w_\lambda) = 0_{\mathbf{L} \circ \mathbf{L}, \mathbf{L}}$$

where  $t_\lambda = (\text{id}_{\mathbf{L}} * \lambda) \circ (\lambda * \text{id}_{\mathbf{L}})$ ,  $w_\lambda = (\text{id}_{\mathbf{L}} * \lambda) \circ (\lambda * \text{id}_{\mathbf{L}})$ , and is such that the following diagram commutes:

$$(23) \quad \begin{array}{ccc} \mathbf{L} \circ \mathbf{L} \circ \mathbf{L} & \xrightarrow{\lambda * \text{id}_{\mathbf{L}}} & \mathbf{L} \circ \mathbf{L} \circ \mathbf{L} \\ \text{id}_{\mathbf{L}} * \Lambda \downarrow & & \downarrow \text{id}_{\mathbf{L}} * \lambda \\ & \mathbf{L} \circ \mathbf{L} \circ \mathbf{L} & \\ & \downarrow \Lambda * \text{id}_{\mathbf{L}} & \\ \mathbf{L} \circ \mathbf{L} & \xrightarrow{\lambda} & \mathbf{L} \circ \mathbf{L} \end{array}$$

$\zeta : (\mathbf{L}, \lambda, \Lambda) \rightarrow (\mathbf{L}', \lambda', \Lambda')$  is a morphism of Lie monads on  $\mathcal{C}$  if  $\zeta : \mathbf{L} \rightarrow \mathbf{L}'$  is a natural transformation satisfying the two following conditions:

- $\Lambda'_X \circ (\zeta * \zeta)_X = \zeta_X \circ \Lambda_X$
- $\lambda'_X \circ (\zeta * \zeta)_X = (\zeta * \zeta)_X \circ \lambda_X$ ,

whenever  $X$  is an object of  $\mathcal{C}$ .

Lie monads and their morphisms form a category, which will be denoted  $\text{LieMnd}(\mathcal{C})$ .

A first elementary example of a Lie monad is the following.

**Example 4.2.** Let  $\mathcal{C}$  be an additive monoidal category and  $(L, \lambda_L, \Lambda_L)$  be a YB-Lie algebra in  $\mathcal{C}$ . Then we have that  $(-\otimes L, \lambda, \Lambda)$  is a Lie monad on  $\mathcal{C}$ , where  $\lambda$  and  $\Lambda$  are defined on any object  $M$  in  $\mathcal{C}$  as follows:

$$\begin{aligned} \lambda_M &:= M \otimes \lambda_L : M \otimes L \otimes L \rightarrow M \otimes L \otimes L \\ \Lambda_M &:= M \otimes \Lambda_L : M \otimes L \otimes L \rightarrow M \otimes L \end{aligned}$$

This is easily checked by using the antisymmetry and Jacobi-identity of  $\Lambda_L$  in  $\mathcal{C}$ . The condition (23) is also satisfied; it is condition (10), combined with the naturality of the associativity constraint  $a$ . Recall from [7, Example 3.10] that the underlying reason for this example to work is that the functor  $\text{End} : \mathcal{C} \rightarrow \text{End}(\mathcal{C})$  sending an object  $X$  in  $\mathcal{C}$  to the endofunctor  $-\otimes X$  is a strong monoidal functor.

This leads to the following.

**Proposition 4.3.** Let  $\mathcal{C}$  be an additive monoidal category. Then there is a functor

$$\underline{\text{Mnd}} : \text{YBLieAlg}(\mathcal{C}) \rightarrow \text{LieMnd}(\mathcal{C}).$$

*Proof.* Let  $(L, \lambda_L, \Lambda_L)$  be a YB-Lie algebra in  $\mathcal{C}$ , then

$$\underline{\text{Mnd}}((L, \lambda_L, \Lambda_L)) = (-\otimes L, \lambda, \Lambda),$$

where  $\lambda$  and  $\Lambda$  are as defined in Example 4.2.

Now, whenever  $f : (L, \lambda_L, \Lambda_L) \rightarrow (L', \lambda_{L'}, \Lambda_{L'})$  is a morphism in  $\text{YBLieAlg}(\mathcal{C})$ ,  $\underline{\text{Mnd}}(f)$  is the natural transformation from  $-\otimes L$  to  $-\otimes L'$ , defined for any object  $X$  in  $\mathcal{C}$  as  $\underline{\text{Mnd}}(f)_X = X \otimes f$ . It is easily verified that this defines a morphism in  $\text{LieMnd}(\mathcal{C})$ , using subsequently the facts that  $f$  preserves the Lie-bracket and the Yang-Baxter operator.  $\square$

The following provides a partial converse of Proposition 4.3.

**Proposition 4.4.** *Let  $\mathcal{C}$  be an additive monoidal category. Suppose that the unit object  $I$  is a regular generator and that the endofunctors  $- \otimes X$  and  $X \otimes -$  preserve colimits in  $\mathcal{C}$  for any object  $X$  in  $\mathcal{C}$ . Let  $L$  be an object in  $\mathcal{C}$ .*

*If  $(- \otimes L, \lambda, \Lambda)$  is a Lie monad on  $\mathcal{C}$  then  $(L, \lambda_I, \Lambda_I)$  is a YB-Lie algebra in  $\mathcal{C}$ .*

*Moreover, there is a bijective correspondence between YB-Lie algebra-structures on  $L$  and Lie monad-structures on the endofunctor  $- \otimes L$ .*

*Proof.* Using the fact that  $I$  is a regular generator, one proves that  $\lambda_I \otimes X \simeq \lambda_{I \otimes X}$  and  $\Lambda_I \otimes X \simeq \Lambda_{I \otimes X}$  for all objects  $X \in \mathcal{C}$ . Applying this fact, one easily verifies the antisymmetry and Jacobi identity for the Lie monad  $- \otimes L$  from the corresponding properties for the YB-Lie algebra  $L$ .

For the last statement, one needs to verify that the construction of Proposition 4.3 together with the construction above leads to the bijective correspondence. This is a typical computation, see e.g. [17, Theorem 1.11] for a similar case.  $\square$

The second class of generating examples of Lie monads is provided by Lie coalgebras.

**Example 4.5.** Let  $\mathcal{C}$  be an additive, left closed monoidal category and  $(C, \gamma_C, \Upsilon_C)$  be a YB-Lie coalgebra in  $\mathcal{C}$ , then  $(H(C, -), \gamma, \Upsilon)$ , with  $\gamma$  and  $\Upsilon$  defined on any object  $M$  by the following diagrams (here the maps  $\Pi_{C,M}^C$  are the isomorphisms from Lemma 1.1)

$$\begin{array}{ccc}
 H(C, H(C, M)) & \xrightarrow{\gamma_M} & H(C, H(C, M)) \\
 \downarrow (\Pi_{C,M}^C)^{-1} & & \uparrow \Pi_{C,M}^C \\
 H(C \otimes C, M) & \xrightarrow{H(\gamma_C, M)} & H(C \otimes C, M) \\
 \\ 
 H(C, H(C, M)) & \xrightarrow{\Upsilon_M} & H(C, M) \\
 \searrow (\Pi_{C,M}^C)^{-1} & & \nearrow H(\Upsilon_C, M) \\
 & H(C \otimes C, M) & 
 \end{array}$$

is a Lie monad. Indeed, it is easily checked that  $H(C, -)$  is additive and the antisymmetry and Jacobi identity for the Lie monad are verified using the corresponding properties of  $\Upsilon_C$  and  $\gamma_C$ .

Similar to Proposition 4.3, we obtain the following

**Proposition 4.6.** *Let  $\mathcal{C}$  be an additive monoidal category. Then there is a functor*

$$\underline{\mathbf{Mnd}}' : \mathbf{YBLieCoAlg}(\mathcal{C}) \rightarrow \mathbf{LieMnd}(\mathcal{C}).$$

*Remark 4.7.* Dually to all Definitions and Theorems above, one can introduce and study Lie comonads on additive categories. All Lie comonads form a category  $\mathbf{LieCoMnd}(\mathcal{C})$ . Without mentioning all details explicitly, let us just mention (as this will be used in the sequel) some notation.

Let  $(C, \gamma_C, \Upsilon_C)$  be a YB-Lie coalgebra in  $\mathcal{C}$ . One has a Lie comonad  $(C \otimes -, \tilde{\gamma}, \tilde{\Upsilon})$ , defined in the obvious way. Letting  $(L, \lambda_L, \Lambda_L)$  be a YB-Lie algebra in  $\mathcal{C}$  and provided  $\mathcal{C}$  is right closed, one also has a Lie comonad  $(H'(L, -), \tilde{\lambda}, \tilde{\Lambda})$ . This then induces functors  $\underline{\mathbf{CMnd}} : \mathbf{YBLieCoAlg}(\mathcal{C}) \rightarrow \mathbf{LieCoMnd}(\mathcal{C})$  and  $\underline{\mathbf{CMnd}}' : \mathbf{YBLieAlg}(\mathcal{C}) \rightarrow \mathbf{LieCoMnd}(\mathcal{C})$ .

**4.2. The Eilenberg-Moore category of a Lie monad.** Let  $(\mathbf{L}, \lambda, \Lambda)$  be a Lie monad on an additive category  $\mathcal{C}$ . We construct the category of *Eilenberg-Moore-Lie* objects  $\mathbf{EML}(\mathbf{L})$  whose objects are couples  $(X, \varrho_X)$ , where  $X$  is an object and  $\varrho_X : \mathbf{L}X \rightarrow X$  is a morphism in  $\mathcal{C}$  such that

$$(\varrho_X \circ \mathbf{L}\varrho_X) \circ (1_{\mathbf{L}\mathbf{L}X} - \lambda_X) - \varrho_X \circ \Lambda_X = 0_{\mathbf{L}\mathbf{L}X, X}.$$

The morphisms of  $\mathbf{EML}(\mathbf{L})$  are morphisms  $f : X \rightarrow Y$  in  $\mathcal{C}$  such that

$$\begin{array}{ccc} \mathbf{L}X & \xrightarrow{\varrho_X} & X \\ \mathbf{L}f \downarrow & & \downarrow f \\ \mathbf{L}Y & \xrightarrow{\varrho_Y} & Y \end{array}$$

The constructions of Lie monads out of YB-Lie algebras as in the previous section correspond nicely with the notion of the Eilenberg-Moore category.

**Proposition 4.8.** *Let  $(L, \lambda_L, \Lambda_L)$  be a YB-Lie algebra in  $\mathcal{C}$ . Then there is a functor*

$$\mathbf{MND}_L : \mathbf{LieMod}(L) \rightarrow \mathbf{EML}(\mathbf{Mnd}(L)),$$

*which is an equivalence of categories.*

*Remark 4.9.* Let  $(\mathbf{C}, \gamma, \Upsilon)$  be a Lie comonad. Then one can introduce in the canonical way the category of Eilenberg-Moore-Lie objects for this Lie-comonad. Furthermore, if  $C$  is a Lie coalgebra, then dually to Proposition 4.8 we find a functor  $\mathbf{CMND}_C : \mathbf{LieCoMod}(C) \rightarrow \mathbf{EML}(\mathbf{CMnd}(C))$ . If  $L$  is a Lie algebra, then we also have a functor  $\mathbf{CMND}'_L : \mathbf{LieMod}(L) \rightarrow \mathbf{EML}(\mathbf{CMnd}'(L))$ .

## 5. DUALITIES BETWEEN LIE ALGEBRAS AND LIE COALGEBRAS

**5.1. Michaelis pairs.** Throughout this section, let  $\mathcal{C}$  be an additive, closed (strict) monoidal category.

**Definition 5.1.** (1) A *Michaelis pair*  $(L, C, \text{ev})$  consists of a YB-Lie algebra  $(L, \lambda_L, \Lambda_L)$ , a YB-Lie coalgebra  $(C, \gamma_C, \Upsilon_C)$  and a morphism

$$\text{ev} : L \otimes C \rightarrow I$$

in  $\mathcal{C}$  that renders commutative the following diagrams

$$(24) \quad \begin{array}{ccccc} L \otimes L \otimes C & \xrightarrow{L \otimes L \otimes \Upsilon_C} & L \otimes L \otimes C \otimes C & \xrightarrow{L \otimes \text{ev} \otimes C} & L \otimes C \\ \Lambda_L \otimes C \downarrow & & & & \downarrow \text{ev} \\ L \otimes C & \xrightarrow{\text{ev}} & I & & \end{array}$$

and

$$(25) \quad \begin{array}{ccccc} L \otimes L \otimes C \otimes C & \xrightarrow{L \otimes L \otimes \gamma_C} & L \otimes L \otimes C \otimes C & \xrightarrow{L \otimes \text{ev} \otimes C} & L \otimes C \\ \lambda_L \otimes C \otimes C \downarrow & & & & \downarrow \text{ev} \\ L \otimes L \otimes C \otimes C & \xrightarrow{L \otimes \text{ev} \otimes C} & L \otimes C & \xrightarrow{\text{ev}} & I \end{array}$$

The morphism  $\text{ev}$  is called a *duality* between  $L$  and  $C$ ; the set of all dualities between  $L$  and  $C$  is denoted by  $\mathbf{Dual}(L, C)$ .

(2) A morphism between two Michaelis pairs is a couple  $(\phi, \psi) : (L, C, \text{ev}) \rightarrow (L', C', \text{ev}')$ , where  $\phi : L \rightarrow L'$  is a YB-Lie algebra morphism and  $\psi : C' \rightarrow C$  is a YB-Lie coalgebra morphism rendering the following diagram commutative

$$(26) \quad \begin{array}{ccc} L \otimes C' & \xrightarrow{L \otimes \psi} & L \otimes C \\ \phi \otimes C' \downarrow & & \downarrow \text{ev} \\ L' \otimes C' & \xrightarrow{\text{ev}'} & I \end{array}$$

(3) Michaelis pairs and their morphisms form a new category  $\underline{\mathbf{Mich}}(\mathcal{C})$ .

Let  $(L, C, \text{ev})$  be a Michaelis pair. Then using the adjunction properties  $\text{Hom}(X \otimes L \otimes C, X) \simeq \text{Hom}(X \otimes L, \text{H}(C, X))$  and  $\text{Hom}(L \otimes C \otimes X, X) \simeq \text{Hom}(C \otimes X, \text{H}'(L, X))$ , we can associate to the Michaelis pair two morphisms that are natural in  $X$ , as follows:

$$(27) \quad \zeta_X = \text{H}(C, X \otimes \text{ev}) \circ \eta_{X \otimes L}^C : X \otimes L \rightarrow \text{H}(C, X)$$

$$(28) \quad \theta_X = \text{H}'(L, \text{ev} \otimes X) \circ \eta_{C \otimes X}^{L'} : C \otimes X \rightarrow \text{H}'(L, X)$$

where  $\eta$  (respectively  $\eta'$ ) denotes -as before- the counit of the adjunction associated the left (respectively right) closedness of  $\mathcal{C}$ . Using notations of the previous section and denoting all Lie monad morphisms  $\text{Hom}(\underline{\text{Mnd}}(L), \underline{\text{Mnd}}'(C))$  and Lie comonad morphisms  $\text{Hom}(\underline{\text{CMnd}}(C), \underline{\text{CMnd}}'(L))$ , we now have the following result:

**Proposition 5.2.** *Let  $(L, \lambda, \Lambda_L)$  be a YB-Lie algebra and  $(C, \gamma, \Upsilon)$  be a YB-Lie coalgebra in  $\mathcal{C}$ .*

(i) *There are maps*

$$\text{Dual}(L, C) \xrightleftharpoons[\beta]{\alpha} \text{Hom}(\underline{\text{Mnd}}(L), \underline{\text{Mnd}}'(C))$$

*such that  $\beta \circ \alpha = 1_{\text{Dual}(L, C)}$ . Moreover, if  $I$  is a regular generator, then  $\alpha \circ \beta = 1_{\text{Hom}(\underline{\text{Mnd}}(L), \underline{\text{Mnd}}'(C))}$ .*

(ii) *There are maps*

$$\text{Dual}(L, C) \xrightleftharpoons[\beta']{\alpha'} \text{Hom}(\underline{\text{CMnd}}(C), \underline{\text{CMnd}}'(L))$$

*such that  $\beta' \circ \alpha' = 1_{\text{Dual}(L, C)}$ . Moreover, if  $I$  is a regular generator, then  $\alpha' \circ \beta' = 1_{\text{Hom}(\underline{\text{CMnd}}(C), \underline{\text{CMnd}}'(L))}$ .*

*Proof.* (i)  $\alpha$ . Let  $\text{ev}$  be a duality between  $L$  and  $C$ . We define  $\alpha(\text{ev}) = \zeta$  as in (27) and use notation as in Example 4.2 and Example 4.5 for  $\underline{\text{Mnd}}(L)$  and  $\underline{\text{Mnd}}'(C)$ . Then  $\zeta$  will be a morphism of Lie monads if and only if for any object  $X \in \mathcal{C}$ ,

$$\Upsilon_X \circ (\zeta * \zeta)_X = \zeta_X \circ \Lambda_X, \quad \gamma_X \circ (\zeta * \zeta)_X = (\zeta * \zeta)_X \circ \lambda_X.$$

We only prove the first identity, the second one follows by a similar computation. We can compute

$$\begin{aligned} \Upsilon_X \circ (\zeta * \zeta)_X &= \text{H}(\Upsilon_C, X) \circ (\Pi_{C, X}^C)^{-1} \circ \text{H}(C, \text{H}(C, X \otimes \text{ev})) \circ \text{H}(C, \eta_{X \otimes L}^C) \circ \text{H}(C, X \otimes L \otimes \text{ev}) \circ \eta_{X \otimes L \otimes L}^C \\ &= \text{H}(\Upsilon_C, X) \circ \text{H}(C \otimes C, X \otimes \text{ev}) \circ (\Pi_{C, X \otimes L \otimes C}^C)^{-1} \circ \text{H}(C, \eta_{X \otimes L}^C) \circ \text{H}(C, X \otimes L \otimes \text{ev}) \circ \eta_{X \otimes L \otimes L}^C \\ &= \text{H}(C, X \otimes \text{ev}) \circ \underbrace{\text{H}(\Upsilon_C, X \otimes L \otimes C) \circ (\Pi_{C, X \otimes L \otimes C}^C)^{-1} \circ \text{H}(C, \eta_{X \otimes L}^C)}_{\text{underbraced part}} \circ \text{H}(C, X \otimes L \otimes \text{ev}) \circ \eta_{X \otimes L \otimes L}^C \end{aligned}$$

Let us first compute the underbraced part separately, then we find, using (4)

$$\begin{aligned} &\text{H}(\Upsilon_C, X \otimes L \otimes C) \circ (\Pi_{C, X \otimes L \otimes C}^C)^{-1} \circ \text{H}(C, \eta_{X \otimes L}^C) \\ &= \text{H}(\Upsilon_C, X \otimes L \otimes C) \circ \text{H}(C \otimes C, \epsilon_{X \otimes L \otimes C}^C \circ (\epsilon_{\text{H}(C, X \otimes L \otimes C)}^C \otimes C)) \circ \eta_{\text{H}(C, \text{H}(C, X \otimes L \otimes C))}^{C \otimes C} \circ \text{H}(C, \eta_{X \otimes L}^C) \\ &= \text{H}(\Upsilon_C, X \otimes L \otimes C) \circ \text{H}(C \otimes C, \epsilon_{X \otimes L \otimes C}^C \circ (\epsilon_{\text{H}(C, X \otimes L \otimes C)}^C \otimes C) \circ (\text{H}(C \otimes C, \eta_{X \otimes L}^C) \otimes C \otimes C)) \circ \eta_{\text{H}(C, X \otimes L)}^{C \otimes C} \\ &= \text{H}(\Upsilon_C, X \otimes L \otimes C) \circ \text{H}(C \otimes C, \epsilon_{X \otimes L \otimes C}^C \circ (\eta_{X \otimes L}^C \otimes C) \circ (\epsilon_{X \otimes L}^C \otimes C)) \circ \eta_{\text{H}(C, X \otimes L)}^{C \otimes C} \\ &= \text{H}(\Upsilon_C, X \otimes L \otimes C) \circ \text{H}(C \otimes C, \epsilon_{X \otimes L}^C \otimes C) \circ \eta_{\text{H}(C, X \otimes L)}^{C \otimes C} \\ &= \text{H}(C, \epsilon_{X \otimes L}^C \otimes C) \circ \text{H}(\Upsilon_C, \text{H}(C, X \otimes L) \otimes C \otimes C) \circ \eta_{\text{H}(C, X \otimes L)}^{C \otimes C} \\ &= \text{H}(C, \epsilon_{X \otimes L}^C \otimes C) \circ \text{H}(C, \text{H}(C, X \otimes L) \otimes \Upsilon_C) \circ \eta_{\text{H}(C, X \otimes L)}^C \end{aligned}$$



All equalities follow by naturality and adjunction property of closedness, in particular the last equality follows by the naturality of  $\eta$  in the upper argument. We can now continue

$$\begin{aligned}
& \Upsilon_X \circ (\zeta * \zeta)_X \\
&= \mathbf{H}(C, X \otimes \text{ev}) \circ \mathbf{H}(C, \epsilon_{X \otimes L}^C \otimes C) \circ \mathbf{H}(C, \mathbf{H}(C, X \otimes L) \otimes \Upsilon_C) \circ \eta_{\mathbf{H}(C, X \otimes L)}^C \circ \mathbf{H}(C, X \otimes L \otimes \text{ev}) \circ \eta_{X \otimes L \otimes L}^C \\
&= \mathbf{H}(C, (X \otimes \text{ev}) \circ (\epsilon_{X \otimes L}^C \otimes C)) \circ (\mathbf{H}(C, X \otimes L) \otimes \Upsilon_C) \circ (\mathbf{H}(C, X \otimes L \otimes \text{ev}) \otimes C) \circ \eta_{\mathbf{H}(C, X \otimes L \otimes L \otimes C)}^C \circ \eta_{X \otimes L \otimes L}^C \\
&= \mathbf{H}(C, (X \otimes \text{ev}) \circ (X \otimes L \otimes \text{ev} \otimes C)) \circ (\epsilon_{X \otimes L \otimes L \otimes C}^C \otimes C) \circ (\mathbf{H}(C, X \otimes L \otimes L \otimes C) \otimes \Upsilon_C) \circ (\eta_{X \otimes L \otimes L}^C \otimes C) \circ \eta_{X \otimes L \otimes L}^C \\
&= \mathbf{H}(C, (X \otimes \text{ev}) \circ (X \otimes L \otimes \text{ev} \otimes C)) \circ (\epsilon_{X \otimes L \otimes L \otimes C}^C \otimes C) \circ (\eta_{X \otimes L \otimes L}^C \otimes C \otimes C) \circ (X \otimes L \otimes L \otimes \Upsilon_C) \circ \eta_{X \otimes L \otimes L}^C \\
&= \mathbf{H}(C, (X \otimes \text{ev}) \circ (X \otimes L \otimes \text{ev} \otimes C) \circ (X \otimes L \otimes L \otimes \Upsilon_C)) \circ \eta_{X \otimes L \otimes L}^C \\
&= \mathbf{H}(C, (X \otimes \text{ev}) \circ (X \otimes \Lambda_L \otimes C)) \circ \eta_{X \otimes L \otimes L}^C \\
&= \mathbf{H}(C, X \otimes \text{ev}) \circ \eta_{X \otimes L}^C \circ (X \otimes \Lambda_L) = \zeta_X \circ \Lambda_X
\end{aligned}$$

(i) $\beta$ . Suppose that  $\zeta : \mathbf{Mnd}(L) \rightarrow \mathbf{Mnd}'(C)$  is a Lie monad morphism. The adjunction property  $\mathbf{Hom}(X \otimes L \otimes C, X) \simeq \mathbf{Hom}(X \otimes L, \mathbf{H}(C, X))$  now allows us to define  $\text{ev} = \epsilon_I^C \circ (\zeta_I \otimes C)$ . Then, by the computations of the first part of the proof, we find from  $\Upsilon_X \circ (\zeta * \zeta)_X = \zeta_X \circ \Lambda_X$  that

$$\mathbf{H}(C, (X \otimes \text{ev}) \circ (X \otimes L \otimes \text{ev} \otimes C) \circ (X \otimes L \otimes L \otimes \Upsilon_C)) \circ \eta_{X \otimes L \otimes L}^C = \mathbf{H}(C, (X \otimes \text{ev}) \circ (X \otimes \Lambda_L \otimes C)) \circ \eta_{X \otimes L \otimes L}^C$$

and therefore

$$(29) \quad (X \otimes \text{ev}) \circ (X \otimes L \otimes \text{ev} \otimes C) \circ (X \otimes L \otimes L \otimes \Upsilon_C) = (X \otimes \text{ev}) \circ (X \otimes \Lambda_L \otimes C)$$

To see this, put  $l$  the left-hand side of (29) and  $r$  the right-hand side, then tensor  $l$  and  $r$  on the right-hand side with the identity morphism on  $C$ , and compose both sides with  $\epsilon_X^C$ . We then obtain

$$\epsilon_X^C \circ (\mathbf{H}(C, l) \otimes C) \circ (\eta_{X \otimes L \otimes L}^C \otimes C) = \epsilon_X^C \circ (\mathbf{H}(C, r) \otimes C) \circ (\eta_{X \otimes L \otimes L}^C \otimes C),$$

which is equivalent to

$$l \circ \epsilon_{X \otimes L \otimes L \otimes C}^C \circ \eta_{X \otimes L \otimes L}^C \otimes C = r \circ \epsilon_{X \otimes L \otimes L \otimes C}^C \circ \eta_{X \otimes L \otimes L}^C \otimes C,$$

which implies (29). If we then take  $X = I$  in (29), we obtain (24). Similarly,  $\gamma_X \circ (\zeta * \zeta)_X = (\zeta * \zeta)_X \circ \Lambda_X$  implies (25).

(i). We still have to check that both constructions above are mutual inverses. So let  $\text{ev}$  be the evaluation map of a given Michaelis pair  $(L, C, \text{ev})$  and denote  $\text{ev}' = \beta \circ \alpha(\text{ev})$ , then we find

$$\begin{aligned}
\text{ev}' &= \epsilon_I^C \circ (\mathbf{H}(C, \text{ev}) \otimes C) \circ (\eta_L^C \otimes C) \\
&= \text{ev} \circ \epsilon_{L \otimes C}^C \circ (\eta_L^C \otimes C) = \text{ev}
\end{aligned}$$

The other way around, suppose that  $I$  is a regular generator. Given a Lie monad morphism  $\zeta$ , we denote  $\zeta' = \alpha \circ \beta(\zeta)$ , such that

$$\begin{aligned}
\zeta'_X &= \mathbf{H}(C, X \otimes (\epsilon_I^C \circ (\zeta_I \otimes C))) \circ \eta_{X \otimes L}^C \\
&= \mathbf{H}(C, X \otimes \epsilon_I^C) \circ \eta_{X \otimes \mathbf{H}(C, I)}^C \circ (X \otimes \zeta_I)
\end{aligned}$$

Hence, clearly  $\zeta'_I = \zeta_I$ . On the other hand,

$$\zeta_X = \mathbf{H}(C, \epsilon_X^C) \circ \eta_{\mathbf{H}(C, X)}^C \circ \zeta_X = \mathbf{H}(C, \epsilon_X^C) \circ \mathbf{H}(C, \zeta_X \otimes L) \circ \eta_{X \otimes L}^C,$$

therefore,  $\zeta_X = \zeta'_X$  if and only if  $\epsilon_X^C \circ (\zeta_X \otimes C) = (X \otimes \epsilon_I) \circ (X \otimes \zeta_I \otimes C)$ . Denote these natural transformations by  $\sigma_X$  and  $\tau_X$  respectively. As  $I$  is a regular generator, we can construct for

any object  $X$  a coequalizer  $(X, q)$  starting from a suitable fork  $I^{(J)} \rightrightarrows I^{(K)}$ . This way we obtain a diagram

$$\begin{array}{ccccc}
 I^{(J)} & \xrightarrow{\quad\quad\quad} & I^{(K)} & \xrightarrow{\quad q \quad} & X \\
 \tau_I^{(J)} \uparrow \uparrow \sigma_I^{(J)} & & \tau_I^{(K)} \uparrow \uparrow \sigma_I^{(K)} & & \tau_X \uparrow \uparrow \sigma_X \\
 I^{(J)} \otimes L \otimes C & \xrightarrow{\quad\quad\quad} & I^{(K)} \otimes L \otimes C & \xrightarrow{\quad q \otimes L \otimes C \quad} & X \otimes L \otimes C
 \end{array}$$

In this diagram both lines are coequalizers (the lower line because  $\mathcal{C}$  is a closed category, hence functor of the form  $- \otimes Y$  have a right adjoint and therefore preserve colimits). By the naturality of  $\sigma$  and  $\tau$ , the diagram commutes serially (i.e. it commutes if we only consider the arrows with  $\tau$  and it commutes if we only consider arrows with  $\sigma$ ) and since  $\sigma_I = \tau_I$  we then find by the universal property of the coequalizer that  $\tau_X = \sigma_X$ .

(ii). Is proven in the same way.  $\square$

**Proposition 5.3.** *Let  $(L, C, \text{ev})$  be a Michaelis pair. Then there is a functor*

$$F : \text{LieCoMod}(C) \rightarrow \text{LieMod}(L),$$

*from the category of left  $C$ -Lie comodules to the category of left  $L$ -Lie modules.*

*Proof.* Let  $(X, \delta^X)$  be a left  $C$ -Lie comodule. Then we define a left Lie-action  $\varrho_X$  on  $X$  by

$$\varrho_X : L \otimes X \xrightarrow{L \otimes \delta^X} L \otimes C \otimes X \xrightarrow{\text{ev} \otimes X} I \otimes X \cong X$$

Let us check that this is indeed a Lie module. We compute

$$\begin{aligned}
 \varrho_X \circ (L \otimes \varrho_X) &= (\text{ev} \otimes X) \circ (L \otimes \delta^X) \circ (L \otimes \text{ev} \otimes X) \circ (L \otimes L \otimes \delta^X) \\
 &= (\text{ev} \otimes X) \circ (L \otimes \text{ev} \otimes C \otimes X) \circ (L \otimes L \otimes C \otimes \delta^X) \circ (L \otimes L \otimes \delta^X)
 \end{aligned}$$

and

$$\begin{aligned}
 \varrho_X \circ (L \otimes \varrho_X) \circ (\lambda \otimes X) &= (\text{ev} \otimes X) \circ (L \otimes \delta^X) \circ (L \otimes \text{ev} \otimes X) \circ (L \otimes L \otimes \delta^X) \circ (\lambda \otimes X) \\
 &= (\text{ev} \otimes X) \circ (L \otimes \text{ev} \otimes C \otimes X) \circ (\lambda \otimes C \otimes C \otimes X) \circ (L \otimes L \otimes C \otimes \delta^X) \circ (L \otimes L \otimes \delta^X) \\
 &= (\text{ev} \otimes X) \circ (L \otimes \text{ev} \otimes C \otimes X) \circ (L \otimes L \otimes \gamma \otimes X) \circ (L \otimes L \otimes C \otimes \delta^X) \circ (L \otimes L \otimes \delta^X)
 \end{aligned}$$

Similarly, we find

$$\begin{aligned}
 \varrho_X \circ (\Lambda \otimes X) &= (\text{ev} \otimes X) \circ (L \otimes \delta^X) \circ (\Lambda \otimes X) \\
 &= (\text{ev} \otimes X) \circ (\Lambda \otimes C \otimes X) \circ (L \otimes L \otimes \delta^X) \\
 &= (\text{ev} \otimes X) \circ (L \otimes \text{ev} \otimes C \otimes X) \circ (L \otimes L \otimes \Upsilon \otimes X) \circ (L \otimes L \otimes \delta^X)
 \end{aligned}$$

Combining these equalities, we find

$$\begin{aligned}
 &\varrho_X \circ (L \otimes \varrho_X) - \varrho_X \circ (L \otimes \varrho_X) \circ (\lambda \otimes X) - \varrho_X \circ (\Lambda \otimes X) \\
 &= (\text{ev} \otimes X) \circ (L \otimes \text{ev} \otimes C \otimes X) \circ (L \otimes L \otimes \gamma \otimes X) \\
 &\quad \circ \left( L \otimes L \otimes ((C \otimes \delta^X) - ((\gamma \otimes X) \circ (C \otimes \delta^X)) - (\Upsilon \otimes X)) \right) \circ (L \otimes L \otimes \delta^X) = 0
 \end{aligned}$$

where we used the Jacobi identity of the  $C$ -Lie comodule  $X$  in the last equality. Hence we can define  $F(X, \delta^X) = (X, \varrho_X)$ . Furthermore, one easily checks that  $F$  is well-defined on morphisms.  $\square$

**Definition 5.5.** A Michaelis pair is called *strong*, if it is isomorphic to an elementary Michaelis pair in the category **Mich**.

The next proposition characterizes strong Michaelis pairs.

**Proposition 5.6.** *There is a bijective correspondence between:*

- (i) *Strong Michaelis pairs  $(L, C, \text{ev})$ ;*
- (ii) *YB-Lie algebras  $L$  such that  $L$  is a right rigid object in  $\mathcal{C}$ ;*
- (iii) *YB-Lie coalgebras  $C$  such that  $C$  is a left rigid object in  $\mathcal{C}$ ;*
- (iv) *Michaelis pairs  $(L, C, \text{ev})$  such that the associated Lie monad morphism  $\zeta = \alpha(\text{ev})$  (see Proposition 5.2) is an isomorphism;*
- (v) *Michaelis pairs  $(L, C, \text{ev})$  such that the associated Lie comonad morphism  $\theta = \alpha'(\text{ev})$  (see Proposition 5.2) is an isomorphism.*

*Proof.* (ii)  $\Rightarrow$  (i). Consider the right dual  $C = L^*$  of  $L$ . This object is defined up to isomorphism in  $\mathcal{C}$ , and by Proposition 5.4, we can endow  $C$  with the structure of a YB-Lie coalgebra such that  $(L, C, \text{ev})$  is a Michaelis pair. Similarly, one proves (iii)  $\Rightarrow$  (i), and the converses are trivial.

(i)  $\Rightarrow$  (iv). Let us prove that  $\xi_X = (\epsilon_X^C \otimes L) \circ (\text{H}(C, X) \otimes \text{coev})$  is an inverse for  $\zeta_X$ .

$$\begin{aligned}
\zeta_X \circ \xi_X &= (\epsilon_X^C \otimes L) \circ (\text{H}(C, X) \otimes \text{coev}) \circ \text{H}(C, X \otimes \text{ev}) \circ \eta_{X \otimes L}^C \\
&= (\epsilon_X^C \otimes L) \circ \text{H}(C, X \otimes \text{ev}) \otimes C \otimes L \circ (\text{H}(C, X \otimes L \otimes C) \otimes \text{coev}) \circ \eta_{X \otimes L}^C \\
&= (X \otimes \text{ev} \otimes L) \circ (\epsilon_{X \otimes L \otimes C} \otimes L) \circ (\eta_{X \otimes L}^C \otimes C \otimes L) \circ (X \otimes L \otimes \text{coev}) \\
&= (X \otimes \text{ev} \otimes L) \circ (X \otimes L \otimes \text{coev}) = X \otimes L \\
\xi_X \circ \zeta_X &= \text{H}(C, X \otimes \text{ev}) \circ \eta_{X \otimes L}^C \circ (\epsilon_X^C \otimes L) \circ (\text{H}(C, X) \otimes \text{coev}) \\
&= \text{H}(C, X \otimes \text{ev}) \circ \text{H}(C, \epsilon_X^C \otimes L \otimes C) \circ \eta_{\text{H}(C, X) \otimes C \otimes L}^C \circ (\text{H}(C, X) \otimes \text{coev}) \\
&= \text{H}(C, \epsilon_X^C) \circ \text{H}(C, \text{H}(C, X) \otimes C \otimes \text{ev}) \circ \text{H}(C, \text{H}(C, X) \otimes \text{coev} \otimes C) \circ \eta_{\text{H}(X, X)}^C \\
&= \text{H}(C, \epsilon_X^C) \circ \eta_{\text{H}(X, X)}^C = \text{H}(C, X)
\end{aligned}$$

where we used the expression for  $\zeta_X$  from (27).

(iv)  $\Rightarrow$  (i). We define  $\text{coev} = \zeta_C^{-1} \circ \eta_I^C$ . Then we find

$$\begin{aligned}
(C \otimes \text{ev}) \circ (\text{coev} \otimes C) &= (C \otimes \text{ev}) \circ (\text{coev} \otimes C) \circ \eta_C^C \circ (\eta_I^C \otimes C) \\
&= \eta_C^C \circ (\text{H}(C, C \otimes \text{ev}) \otimes C) \circ (\text{H}(C, \text{coev} \otimes C) \otimes C) \circ (\eta_I^C \otimes C) \\
&= \eta_C^C \circ (\text{H}(C, C \otimes \text{ev}) \otimes C) \circ (\text{H}(C, \zeta_C^{-1} \otimes C) \otimes C) \circ (\text{H}(C, \eta_I^C \otimes C) \otimes C) \circ (\eta_I^C \otimes C) \\
&= \eta_C^C \circ (\text{H}(C, C \otimes \text{ev}) \otimes C) \circ (\text{H}(C, \zeta_C^{-1} \otimes C) \otimes C) \circ (\eta_{\text{H}(C, C)} \otimes C) \circ (\eta_I^C \otimes C) \\
&= \eta_C^C \circ (\text{H}(C, C \otimes \text{ev}) \otimes C) \circ (\eta_{C \otimes L}^C \otimes C) \circ (\zeta_C^{-1} \otimes C) \circ (\eta_I^C \otimes C) \\
&= \eta_C^C \circ (\zeta_C \otimes C) \circ (\zeta_C^{-1} \otimes C) \circ (\eta_I^C \otimes C) = C
\end{aligned}$$

A similar computation shows that  $(\text{ev} \otimes L) \circ (L \otimes \text{coev}) = L$ .

(i)  $\Leftrightarrow$  (v) is similar to (i)  $\Leftrightarrow$  (iv). □

**Proposition 5.7.** *Let  $(L, C, \text{ev})$  be a strong Michaelis pair. Then the functor*

$$F : \text{LieCoMod}(C) \rightarrow \text{LieMod}(L)$$

*from Proposition 5.3 is an equivalence of categories.*

*Proof.* We define a functor  $G : \text{LieMod}(L) \rightarrow \text{LieCoMod}(C)$  as follows. Take any left  $L$ -lie module  $(X, \varrho_X)$ . Then we define a  $C$ -Lie coaction  $\delta^X$  on  $C$  by

$$\delta^X : X \xrightarrow{\text{coev} \otimes X} C \otimes L \otimes X \xrightarrow{C \otimes \varrho_X} C \otimes X.$$

One proves similarly as in Proposition 5.3 that  $G$  is well-defined.

Next, we observe that  $FG(X, \varrho_X) \cong (X, \varrho_X)$ . Indeed, if we denote  $FG(X, \varrho_X) = (X, \varrho'_X)$  then

$$\begin{aligned} \varrho'_X &= (\text{ev} \otimes X) \circ (L \otimes C \otimes \varrho_X) \circ (L \otimes \text{coev} \otimes X) \\ &= \varrho_X \circ (\text{ev} \otimes L \otimes X) \circ (L \otimes \text{coev} \otimes X) = \varrho_X \end{aligned}$$

Similarly,  $GF(X, \delta^X) \cong (X, \delta^X)$  and  $(F, G)$  is an equivalence of categories.  $\square$

From Proposition 5.4 and Proposition 5.7, we now immediately have the following result, which is the “Lie version” of the classical analogous result for usual monads (see e.g. [6]).

**Corollary 5.8.** *Let  $(L, R)$  be an adjoint pair of additive endofunctors on an additive category  $\mathcal{A}$ . Then  $L$  is a Lie monad if and only if  $R$  is a Lie comonad, and in this situation the Eilenberg-Moore categories are equivalent.*

*Remark 5.9.* It is an interesting question to ask whether the above study of strong dualities between Lie algebras and Lie coalgebras can be generalized to a more general setting, introducing “rationality” for Lie coalgebras and considering non-degenerate evaluation morphisms.

**Example 5.10** (Finite dimensional Lie algebras). If  $L$  is a finite dimensional  $k$ -Lie algebra, then  $L$  is a (left and right) rigid object in the symmetric monoidal category of  $k$ -vector spaces. Hence  $C = L^*$ , the vectorspace dual of  $L$  is a Lie coalgebra, as we already remarked in Example 3.3(4), and  $(L, C, \text{ev})$  is a strong Michaelis pair, where  $\text{ev}$  is the usual evaluation map. In this situation  $\text{coev}$  is given by the dual basis.

**Example 5.11** (Infinite dimensional Lie algebras). If  $L$  is an infinite dimensional  $k$ -Lie algebra, then  $L$  is no longer a rigid object in  $\mathbf{Vect}(k)$ . However, we can still consider the associated Lie monad  $- \otimes L$ , as a YB-Lie algebra in the monoidal category of endofunctors on  $\mathbf{Vect}(k)$ . As the functor  $- \otimes L$  has a right adjoint  $\mathbf{Hom}_k(L, -)$ , the functor  $- \otimes L$  is right rigid in the category of endofunctors. Hence Proposition 5.6 applies and we find that  $\mathbf{Hom}_k(L, -)$  is a Lie comonad and  $(- \otimes L, \mathbf{Hom}_k(L, -))$  is a strong Michaelis pair in the category of endofunctors. Consequently, by Corollary 5.8 the category of representations of the Lie algebra  $L$  is equivalent with the category of Lie comodules over the Lie comonad  $\mathbf{Hom}_k(L, -)$ . This infinite dimensional example motivates the transition to Lie monads and Lie comonads (hence also YB-Lie algebras, as the category of endofunctors is no longer symmetric).

## 6. DUALITIES BETWEEN LIE ALGEBRAS AND HOPF ALGEBRAS

**6.1. YB-Lie algebra of primitive elements.** In this section,  $\mathcal{C}$  is an additive, monoidal category in which equalizers and coequalizers are preserved by functors of the form  $- \otimes X$  and  $X \otimes -$ , for any object  $X$  in  $\mathcal{C}$ . For the remaining part of this section, we fix a braided bialgebra  $H$  in  $\mathcal{C}$ , in the sense of [16]. More precisely, we consider a 6-tuple  $(H, \mu, \eta, \Delta, \epsilon, \lambda)$  satisfying the following conditions:

- $(H, \mu, \eta)$  is an algebra in  $\mathcal{C}$ ;
- $(H, \Delta, \epsilon)$  is a coalgebra in  $\mathcal{C}$ ;
- $\lambda$  is a YB-operator of order two for  $H$  (this condition is more restrictive than the usual one of [16]);
- The morphism  $\lambda$  is compatible with  $\mu$  in the sense of (15), and in a similar way with  $\eta, \Delta$  and  $\epsilon$ ;

- $\epsilon : H \rightarrow I$  is an algebra morphism;  $\eta : I \rightarrow H$  is a coalgebra morphism in  $\mathcal{C}$  and the following diagram commutes:

$$(30) \quad \begin{array}{ccc} H \otimes H & \xrightarrow{\Delta \otimes \Delta} & H \otimes H \otimes H \otimes H \\ \downarrow \mu & & \downarrow H \otimes \lambda \otimes H \\ & & H \otimes H \otimes H \otimes H \\ & & \downarrow \mu \otimes \mu \\ H & \xrightarrow{\Delta} & H \otimes H \end{array}$$

**Definition 6.1.** The *primitive elements* of  $H$  are defined as the equalizer  $(P(H), \text{eq})$  in the following diagram

$$P(H) \xrightarrow{\text{eq}} H \xrightleftharpoons[\eta \otimes H + H \otimes \eta]{\Delta} H \otimes H.$$

Let us search for a YB-operator (of order two)  $\lambda_{P(H)}$  for  $P(H)$ . Such a morphism  $\lambda_{P(H)} : P(H) \otimes P(H) \rightarrow P(H) \otimes P(H)$  will be constructed out of the commutativity of the following diagrams:

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\Delta \otimes \Delta} & H \otimes H \otimes H \otimes H \\ \lambda_H \downarrow & & \lambda_{H \otimes H} \downarrow \\ H \otimes H & \xrightarrow{\Delta \otimes \Delta} & H \otimes H \otimes H \otimes H \end{array} \quad \begin{array}{ccc} H \otimes H & \xrightarrow{\alpha} & H \otimes H \otimes H \otimes H \\ \lambda_H \downarrow & & \lambda_{H \otimes H} \downarrow \\ H \otimes H & \xrightarrow{\alpha} & H \otimes H \otimes H \otimes H \end{array}$$

where  $\lambda_{H \otimes H} = (H \otimes \lambda_H \otimes H) \circ (\lambda_H \otimes \lambda_H) \circ (H \otimes \lambda_H \otimes H)$  and  $\alpha = (\eta \otimes H + H \otimes \eta) \otimes (\eta \otimes H + H \otimes \eta)$ . Indeed, since  $(P(H) \otimes P(H), \text{eq} \otimes \text{eq})$  is again an equalizer in  $\mathcal{C}$ , the commutativity of the above diagrams implies the existence of a unique morphism  $\lambda_{P(H)} : P(H) \otimes P(H) \rightarrow P(H) \otimes P(H)$  such that

$$(31) \quad (\text{eq} \otimes \text{eq}) \circ \lambda_{P(H)} = \lambda_H \circ (\text{eq} \otimes \text{eq}),$$

by the universal property in the definition of equalizer. So, let us check that these two diagrams commute. We start with the diagram on the left:

$$\begin{aligned} & (H \otimes \lambda_H \otimes H) \circ (\lambda_H \otimes \lambda_H) \circ (H \otimes \lambda_H \otimes H) \circ \Delta \otimes \Delta \\ = & (\Delta \otimes H) \otimes (H \otimes \lambda_H) \otimes (H \otimes \Delta \otimes \Delta) \circ (\lambda_H \otimes \lambda_H) \circ (H \otimes \lambda_H \otimes H) \\ = & (\Delta \otimes H) \otimes (H \otimes \lambda_H) \circ (\lambda_H \otimes H) \otimes (\Delta \otimes H \otimes H) \\ = & \lambda_H \circ (H \otimes \Delta) \otimes (\Delta \otimes H \otimes H) \end{aligned}$$

All of these three equalities use the fact that  $\Delta$  is compatible with  $\lambda_H$ .

We now consider the diagram on the right hand side. Let us check that  $(\eta \otimes H \otimes \eta \otimes H) \circ \lambda_H = \lambda_{H \otimes H} \circ (\eta \otimes H \otimes \eta \otimes H)$ :

$$\begin{aligned} & (H \otimes \lambda_H \otimes H) \circ (\lambda_H \otimes \lambda_H) \circ (H \otimes \lambda_H \otimes H) \circ (\eta \otimes H \otimes \eta \otimes H) \\ = & (H \otimes \lambda_H \otimes H) \circ (\lambda_H \otimes \lambda_H) \circ (\eta \otimes \eta \otimes H \otimes H) \\ = & (H \otimes \lambda_H \otimes H) \circ (\eta \otimes \eta \otimes H \otimes H) \circ \lambda_H \\ = & (\eta \otimes H \otimes \eta \otimes H) \circ \lambda_H \end{aligned}$$

All of these equalities use the compatibility of  $\eta$  with  $\lambda_H$ . Similar computations are made for the other three components of  $\alpha$ .

**Lemma 6.2.**  $\lambda_{P(H)}$  is a YB-operator of order two for  $P(H)$ .

*Proof.* We have, by the universal property of the equalizer, that

$$(\text{eq} \otimes \text{eq}) \circ \lambda_{P(H)} \circ \lambda_{P(H)} = \lambda_H \circ \lambda_H \circ (\text{eq} \otimes \text{eq}) = \text{eq} \otimes \text{eq}.$$

In the second equality we use that  $\lambda_H$  is of order two. Since  $\text{eq} \otimes \text{eq}$  is a monomorphism in  $\mathcal{C}$ , it follows that  $\lambda_{P(H)} \circ \lambda_{P(H)} = P(H) \otimes P(H)$ .

We also have that

$$\begin{aligned} & (\text{eq} \otimes \text{eq} \otimes \text{eq}) \circ (\lambda_{P(H)} \otimes P(H)) \circ (P(H) \otimes \lambda_{P(H)}) \circ (\lambda_{P(H)} \otimes P(H)) \\ &= (\lambda_H \otimes H) \circ (H \otimes \lambda_H) \circ (\lambda_H \otimes H) \circ (\text{eq} \otimes \text{eq} \otimes \text{eq}) \\ &= (H \otimes \lambda_H) \circ (\lambda_H \otimes H) \circ (H \otimes \lambda_H) \circ (\text{eq} \otimes \text{eq} \otimes \text{eq}) \\ &= (\text{eq} \otimes \text{eq} \otimes \text{eq}) \circ (P(H) \otimes \lambda_{P(H)}) \circ (\lambda_{P(H)} \otimes P(H)) \circ (P(H) \otimes \lambda_{P(H)}) \end{aligned}$$

In the first and third equality we use the universal property, whereas in the second one, we use the fact that  $\lambda(H)$  is a YB-operator for  $H$ . Since  $\text{eq} \otimes \text{eq} \otimes \text{eq}$  is also a monomorphism, we find that  $\lambda_{P(H)}$  is a YB-operator as well.  $\square$

A braided Hopf algebra  $H$  in  $\mathcal{C}$  is in particular a YB-algebra in  $\mathcal{C}$ . Hence, applying the functor  $\mathcal{L} : \text{YBA}(\mathcal{C}) \rightarrow \text{YBLieAlg}(\mathcal{C})$  of Construction 2.16 it follows that  $\Lambda_H = \mu \circ (H \otimes H - \lambda_H)$  determines a YB-Lie algebra structure on  $H$ .

We now wish to construct a Lie-bracket  $\Lambda_{P(H)}$  for  $P(H)$ , inherited from the bracket  $\Lambda_H$  we have for  $H$ . This is done very similarly to the construction of  $\lambda_{P(H)}$ , as described above; by universal property-arguments and using the compatibility conditions of  $H$ , together with (30), one verifies the existence of a unique morphism  $\Lambda_{P(H)} : P(H) \otimes P(H) \rightarrow P(H)$  such that

$$(32) \quad \text{eq} \circ \Lambda_{P(H)} = \Lambda_H \circ (\text{eq} \otimes \text{eq}).$$

Moreover, using the fact that  $\Lambda_H$  is a Lie-bracket for  $H$  and keeping in mind that  $\text{eq} \otimes \text{eq}$  and  $\text{eq} \otimes \text{eq} \otimes \text{eq}$  are both monomorphisms, one shows, in a similar fashion as before, that the conditions (8), (9) and (10) are satisfied for  $\Lambda_{P(H)}$ .

Analogously, we can consider  $(Q(H), \text{coeq})$ , the “indecomposables” of  $H$ , to be the coequalizer in the following diagram in  $\mathcal{C}$ :

$$H \otimes H \xrightleftharpoons[H \otimes \epsilon + \epsilon \otimes H]{\mu} H \xrightarrow{\text{coeq}} Q(H).$$

Summarizing, we have the following

**Proposition 6.3.** *Let  $H$  be a braided bialgebra in  $\mathcal{C}$ , then*

- (i)  $(P(H), \lambda_{P(H)}, \Lambda_{P(H)})$  is a YB-Lie algebra in  $\mathcal{C}$  and  $\text{eq} : P(H) \rightarrow \mathcal{L}(H)$  is a YB-Lie algebra morphism;
- (ii)  $(Q(H), \gamma_{Q(H)}, \Gamma_{Q(H)})$  is a YB-Lie coalgebra in  $\mathcal{C}$  and  $\text{coeq} : Q(H) \rightarrow \mathcal{L}^c(H)$  is a YB-Lie coalgebra morphism.

*Proof.* The first statement clearly follows from the discussion above. To see that the second statement holds, we need the existence of a YB-operator  $\gamma_{Q(H)}$  for  $Q(H)$  such that

$$(33) \quad \gamma_{Q(H)} \circ (\text{coeq} \otimes \text{coeq}) = (\text{coeq} \otimes \text{coeq}) \circ \gamma_{Q(H)}$$

and a morphism  $\Gamma_{Q(H)} : Q(H) \rightarrow Q(H) \otimes Q(H)$  such that

$$(34) \quad \text{coeq} \otimes \text{coeq} \circ \Gamma_H = \Gamma_{Q(H)} \circ \text{coeq},$$

where  $\Gamma_H = (H \otimes H - \lambda_{H,H}) \circ \Delta_H$ , the co-bracket for  $H$ . For these ingredients to exist and to satisfy the conditions of Definition 3.1, it is sufficient to perform the construction of primitive elements  $P(-)$  in the opposite category  $\mathcal{C}^{op}$  and remark that bialgebras are “selfdual” objects in a monoidal category, hence bialgebras in  $\mathcal{C}^{op}$ .  $\square$

*Remark 6.4.* When  $\mathcal{C}$  is the category of  $k$ -vectorspaces over a field  $k$ , the coequalizer  $(Q(H), \text{coeq})$  coincides with Michaelis' original definition of  $Q(H)$ , as we remarked in Example 3.3(2).

**6.2. Takeuchi pairs.** Let  $(H, \mu_H, \eta_H, \Delta_H, \epsilon_H, \lambda_H)$  and  $(K, \mu_K, \eta_K, \Delta_K, \epsilon_K, \lambda_K)$  be two braided bialgebras in  $\mathcal{C}$ . We adapt the definition of “dual pair of bialgebras” (cf. [8] e.g.) to the actual setting, embodied by the following definition:

**Definition 6.5.** (1)  $(H, K, \diamond)$  is called a *Takeuchi pair* in  $\mathcal{C}$  if there exists a morphism in  $\mathcal{C}$   $\diamond : H \otimes K \rightarrow I$ , such that the following conditions hold:

- (a)  $\diamond \circ (H \otimes \eta_K) = \epsilon_H$ ;
- (b)  $\diamond \circ (\eta_H \otimes K) = \epsilon_K$ ;
- (c)  $\diamond \circ (\mu_H \otimes K) = \diamond \circ (H \otimes \diamond \otimes K) \circ (H \otimes H \otimes \Delta_K)$ ;
- (d)  $\diamond \circ (H \otimes \mu_K) = \diamond \circ (H \otimes \diamond \otimes K) \circ (\Delta_H \otimes K \otimes K)$ ;
- (e)  $\diamond \circ (H \otimes \diamond \otimes K) \circ (\lambda_H \otimes K \otimes K) = \diamond \circ (H \otimes \diamond \otimes K) \circ (H \otimes H \otimes \lambda_K)$ .

(2) A morphism of Takeuchi pairs is a pair  $(\phi, \psi) : (H, K, \diamond) \rightarrow (H', K', \diamond')$ , where  $\phi : H \rightarrow H'$  and  $\psi : K \rightarrow K'$  are morphisms of braided bialgebras such that  $\diamond' = \diamond \circ (\phi \otimes \psi)$ .

(3) Takeuchi pairs and their morphisms constitute a category that we denote by  $\underline{\text{Tak}}(\mathcal{C})$ .

**Lemma 6.6.** *Let  $(H, K, \diamond)$  be a Takeuchi pair in  $\mathcal{C}$ , then we have the following equality:*

$$\diamond \circ (\Lambda_H \otimes K) = \diamond \circ (H \otimes \diamond \otimes K) \circ (H \otimes H \otimes \Gamma_K),$$

where  $\Gamma_K = (K \otimes K - \lambda_K) \circ \Delta_K$  and  $\Lambda_H = \mu_H \circ (H \otimes H - \lambda_H)$ .

*Proof.* We compute

$$\begin{aligned} & \diamond \circ (\Lambda_H \otimes K) \\ &= \diamond \circ (\mu_H \circ (H \otimes H - \lambda_H) \otimes K) = \diamond \circ (\mu_H \otimes K) \circ ((H \otimes H - \lambda_H) \otimes K) \\ &= \diamond \circ (H \otimes \diamond \otimes K) \circ (H \otimes H \otimes \Delta_K) \circ ((H \otimes H - \lambda_H) \otimes K) \\ &= \diamond \circ (H \otimes \diamond \otimes K) \circ ((H \otimes H - \lambda_H) \otimes K \otimes K) \circ (H \otimes H \otimes \Delta_K) \\ &= \diamond \circ (H \otimes \diamond \otimes K) \circ (H \otimes H \otimes (K \otimes K - \lambda_K)) \circ (H \otimes H \otimes \Delta_K) \\ &= \diamond \circ (H \otimes \diamond \otimes K) \circ (H \otimes H \otimes \Gamma_K) \end{aligned}$$

We used the third condition of Definition 6.5 in the third equality and the fifth condition of Definition 6.5 in the third one.  $\square$

**Proposition 6.7.** *Let  $(H, K, \diamond)$  be a Takeuchi pair in  $\mathcal{C}$ , then  $(P(H), Q(K), \text{ev})$  is a Michaelis pair. Moreover, we obtain a functor*

$$\mathcal{P} : \underline{\text{Tak}}(\mathcal{C}) \rightarrow \underline{\text{Mich}}(\mathcal{C}), \quad \mathcal{P}(H, K, \diamond) = (P(H), Q(K), \text{ev})$$

*Proof.* In order to make our notation not too heavy, let us put  $P = P(H)$  and  $Q = Q(K)$  in what follows. Let us first look for a suitable morphism  $\text{ev} : P \otimes Q \rightarrow I$ . We know that  $(Q, \text{coeq}_K)$  is a coequalizer, and as coequalizers are preserved by tensoring in  $\mathcal{C}$ ,  $(P \otimes Q, P \otimes \text{coeq}_K)$  is a coequalizer as well.

$$\begin{array}{ccccc} P \otimes K \otimes K & \xrightarrow[\quad P \otimes (K \otimes \epsilon_K + \epsilon_K \otimes K) \quad]{P \otimes \mu_K} & P \otimes K & \xrightarrow{P \otimes \text{coeq}_K} & P \otimes Q \\ & & & \searrow \scriptstyle \diamond \circ (\text{eq}_H \otimes K) & \downarrow \scriptstyle \text{ev} \\ & & & & I \end{array}$$

Therefore, if  $\diamond \circ (\text{eq}_H \otimes K) : P \otimes K \rightarrow I$  coequalizes the pair  $(P \otimes \mu_K, P \otimes (K \otimes \epsilon_K + \epsilon_K \otimes K))$ , then the universal property induces a (unique) morphism  $\text{ev} : P \otimes Q \rightarrow I$  such that

$$(35) \quad \text{ev} \circ (P \otimes \text{coeq}) = \diamond \circ (\text{eq} \otimes K)$$



We calculate:

$$\begin{aligned}
& \diamond \circ (\text{eq} \otimes K) \circ (P \otimes \mu_K) = \diamond \circ (P \otimes \mu) \circ (\text{eq} \otimes K \otimes K) \\
&= \diamond \circ (H \otimes \diamond \otimes K) \circ (\Delta_H \otimes K \otimes K) \circ (\text{eq} \otimes K \otimes K) \\
&= \diamond \circ (H \otimes \diamond \otimes K) \circ ((\eta_H \otimes H + H \otimes \eta_H) \otimes K \otimes K) \circ (\text{eq} \otimes K \otimes K) \\
&= \diamond \circ (H \otimes (K \otimes \epsilon_K + \epsilon_K \otimes K)) \circ (\text{eq} \otimes K \otimes K) \\
&= \diamond \circ (\text{eq} \otimes K) \circ (P \otimes (\epsilon_K \otimes K + K \otimes \epsilon_K)),
\end{aligned}$$

where we use the fourth condition of Definition 6.5 in the second equality, the definition of the equalizer  $(P, \text{eq})$  in the third equality, and the second condition of Definition 6.5 in the fourth equality.

We now have to prove that the two diagrams, occuring in Definition 5.1, commute. Let us start with the proof of the equality

$$(36) \quad \text{ev} \circ (P \otimes \text{ev} \otimes Q) \circ (P \otimes P \otimes \gamma_Q) = \text{ev} \circ (P \otimes \text{ev} \otimes Q) \circ (\lambda_P \otimes Q \otimes Q)$$

Applying (33) in the first equality, (35) in the second and sixth one, the fifth condition of Definition 6.5 in the fourth one and (31) in the fifth equality, we find:

$$\begin{aligned}
& \text{ev} \circ (P \otimes \text{ev} \otimes Q) \circ (P \otimes P \otimes \gamma_Q) \circ (P \otimes P \otimes \text{coeq} \otimes \text{coeq}) \\
&= \text{ev} \circ (P \otimes \text{ev} \otimes Q) \circ (P \otimes P \otimes \text{coeq} \otimes \text{coeq}) \circ (P \otimes P \otimes \lambda_K) \\
&= \diamond \circ (H \otimes \diamond \otimes K) \circ (\text{eq} \otimes \text{eq} \otimes K \otimes K) \circ (P \otimes P \otimes \lambda_K) \\
&= \diamond \circ (H \otimes \diamond \otimes K) \circ (H \otimes H \otimes \lambda_K) \circ (\text{eq} \otimes \text{eq} \otimes K \otimes K) \\
&= \diamond \circ (H \otimes \diamond \otimes K) \circ (\lambda_H \otimes K \otimes K) \circ (\text{eq} \otimes \text{eq} \otimes K \otimes K) \\
&= \diamond \circ (H \otimes \diamond \otimes K) \circ (\text{eq} \otimes \text{eq} \otimes K \otimes K) \circ (\lambda_P \otimes K \otimes K) \\
&= \text{ev} \circ (P \otimes \text{ev} \otimes Q) \circ (P \otimes P \otimes \text{coeq} \otimes \text{coeq}) \circ (\lambda_P \otimes K \otimes K) \\
&= \text{ev} \circ (P \otimes \text{ev} \otimes Q) \circ (\lambda_P \otimes K \otimes K) \circ (P \otimes P \otimes \text{coeq} \otimes \text{coeq})
\end{aligned}$$

As  $P \otimes P \otimes \text{coeq} \otimes \text{coeq}$  is an epimorphism in  $\mathcal{C}$ , (36) holds.

We now proceed with proving the commutativity of the other diagram. Using (35) in the second equality and the sixth one, (32) in the third equality, Lemma 6.6 in the fourth one, and finally (34) in the last equality, we calculate consequently:

$$\begin{aligned}
& \text{ev} \circ (\Lambda_P \otimes Q) \circ (P \otimes P \otimes \text{coeq}) = \text{ev} \circ (P \otimes \text{coeq}) \circ (\Lambda_P \otimes K) \\
&= \diamond \circ (\text{eq} \otimes K) \circ (\Lambda_P \otimes K) = \diamond \circ (\Lambda_H \otimes K) \circ (\text{eq} \otimes \text{eq} \otimes K) \\
&= \diamond \circ (H \otimes \diamond \otimes K) \circ (H \otimes H \otimes \Gamma_K) \circ (\text{eq} \otimes \text{eq} \otimes K) \\
&= \diamond \circ (H \otimes \diamond \otimes K) \circ (\text{eq} \otimes \text{eq} \otimes K \otimes K) \circ (P \otimes P \otimes \Gamma_K) \\
&= \text{ev} \circ (P \otimes \text{ev} \otimes Q) \circ (P \otimes P \otimes \text{coeq} \otimes \text{coeq}) \circ (P \otimes P \otimes \Gamma_K) \\
&= \text{ev} \circ (P \otimes \text{ev} \otimes Q) \circ (P \otimes P \otimes \Gamma_Q) \circ (P \otimes P \otimes \text{coeq})
\end{aligned}$$

As  $P \otimes P \otimes \text{coeq}$  is an epimorphism in  $\mathcal{C}$ , the above is equivalent with the equality we were looking for. This establishes the result.  $\square$

**Example 6.8.** Let  $H$  be a Hopf  $k$ -algebra over a field  $k$ , and  $H^\circ$  its Sweedler dual. Denote by  $H'$  the opposite-co-opposite Hopf  $k$ -algebra of  $H^\circ$ . Then  $(H', H, \diamond)$  is a Takeuchi pair, where  $\diamond$  is the usual evaluation map. Hence, we find that  $(P(H'), Q(H), \text{ev})$  is a Michaelis pair, where  $\text{ev}$  is again the usual evaluation map. Michaelis [13] proved moreover that  $P(H^\circ) \cong Q(H)^*$ , i.e. this Michaelis pair is always strong. We generalize this result in a forthcoming paper.

Given a braided Hopf algebra, recall from Section 2.3 that there exists an induction functor  $\text{Ind} : \text{Mod}(H) \rightarrow \text{LieMod}(\mathcal{L}(H))$ . On the other hand, the YB-Lie algebra morphism  $\text{eq} : P(H) \rightarrow H$  induces a functor  $\text{LieMod}(\mathcal{L}(H)) \rightarrow \text{LieMod}(P(H))$ . Therefore, we obtain a combined functor

$$\text{Mod}(H) \rightarrow \text{LieMod}(P(H))$$

Dually, for another braided Hopf algebra  $K$ , we find a functor  $\text{CoMod}(K) \rightarrow \text{LieCoMod}(Q(K))$ . Therefore, given a Takeuchi pair  $(H, K, \diamond)$  we obtain the following diagram of functors between categories of left (Lie) (co) modules.

$$(37) \quad \begin{array}{ccc} \text{CoMod}(K) & \xrightarrow{F} & \text{Mod}(H) \\ \downarrow G' & & \downarrow G \\ \text{LieCoMod}(Q(K)) & \xrightarrow{F'} & \text{LieMod}(P(H)) \end{array}$$

**Theorem 6.9.** *Let  $(L, K, \diamond)$  be a Takeuchi pair, then the diagram of functors (37) commutes.*

*Proof.* Consider a  $K$ -comodule  $(M, \rho^{M,K})$ . Then we have  $F(M, \rho^{M,K}) = (M, \rho_{M,H})$ , where

$$\rho_{M,H} : H \otimes M \xrightarrow{H \otimes \rho^{M,K}} H \otimes K \otimes M \xrightarrow{\diamond \otimes M} M$$

Next, we find  $GF(M, \rho^{M,K}) = G(M, \rho_{M,H}) = (M, \varrho_{M,P(H)})$ , with

$$\varrho_{M,P(H)} : P(M) \otimes M \xrightarrow{\text{eq} \otimes M} H \otimes M \xrightarrow{\rho_{M,H}} M$$

On the other hand, we obtain  $G'(M, \rho^{M,K})$ , where

$$\delta^{M,Q(K)} : M \xrightarrow{\rho^{M,K}} K \otimes M \xrightarrow{\text{coeq} \otimes M} Q(K) \otimes M$$

We continue and compute  $F'G'(M, \rho^{M,K}) = F'(M, \delta^{M,Q(K)}) = (M, \varrho'_{M,P(H)})$  given by

$$\varrho'_{M,P(H)} : P(H) \otimes M \xrightarrow{P(H) \otimes \delta^{M,Q(K)}} P(H) \otimes Q(K) \otimes M \xrightarrow{\text{ev} \otimes M} M$$

Finally, to see that  $F'G' = GF$  it suffices to verify that  $\varrho'_{M,P(H)} = \varrho_{M,P(H)}$ , i.e.

$$\begin{aligned} \varrho'_{M,P(H)} &= (\text{ev} \otimes M) \circ (P(H) \otimes \text{coeq} \otimes M) \circ (P(H) \otimes \rho^{M,K}) \\ &= (\diamond \otimes M) \circ (\text{eq} \otimes K \otimes M) \circ (P(H) \otimes \rho^{M,K}) \\ &= (\diamond \otimes M) \circ (H \otimes \rho^{M,K}) \circ (\text{eq} \otimes M) = \varrho_{M,P(H)} \end{aligned}$$

This finishes the proof.  $\square$

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